Boolean Connectives

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Agenda

- Chapter 1 introduced basic FOL
  (one main aim of book)
- Chapter 2 introduced notion of logical consequence
  (other main aim of book)
- Chapter 3 introduces more features of FOL

Boolean Connectives

Recall that an atomic sentence is a predicate applied to one or more terms:

`Older(father(max),max)`

We now extend FOL with the boolean connectives:

- and, to be written $\wedge$
- or, to be written $\vee$
- not, to be written $\neg$.

Negation ("not")

Truth table:

P	$\neg P$
true	false
false	true

- Symbol $\neg$ is not standard (cf. p. 91);
  in emails and on the web I'll write ~.
- $\neg\neg P$ is equivalent to P
  unlike English, where double negation emphasizes:
  it doesn't make no difference; *there will be no nothing*
- $\neg$`LeftOf(a, b)` is not equivalent to `RightOf(a, b)`

Conjunction ("and")

P	Q	$P \wedge Q$
true	true	true
true	false	false
false	true	false
false	false	false

- in emails and on the web I may write /\ or ^
- English sentences translated using $\wedge$ may
  - not use "and"

    Max is a tall man `Tall(max)` $\wedge$ `Man(max)`

  - carry temporal implications

    Max went home and went to sleep

  - be expressed using other connectives

    Max was home but Claire was not

Disjunction ("or")

P	Q	$P \vee Q$
true	true	true
true	false	true
false	true	true
false	true	false

- in emails and on the web I may write \/ or v.
- the interpretation is "inclusive", not "exclusive": true $\vee$ true $=$ true.
- In English, the default is often "exclusive", as when a waiter offers *soup or salad*
- We can express exclusive or (p. 75):

Disjunction ("or")

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- In English, the default is often "exclusive", as when a waiter offers *soup or salad*
- We can express exclusive or (p. 75): $(P \vee Q) \wedge \neg(P \wedge Q)$
- We can also encode "neither nor": $\neg(P \vee Q)$

Sentences

A sentence P is thus given by

- if P is an atomic sentence then P is also a sentence;
- if P_1 and P_2 are sentences then $P_1 \wedge P_2$ is a sentence;
- if P_1 and P_2 are sentences then $P_1 \vee P_2$ is a sentence;
- if P is a sentence then $\neg P$ is a sentence.

This can be written in "Backus-Naur" notation:

$$\begin{array}{rcl} P & ::= & \text{atomic sentence} \\ & | & P \wedge P \\ & | & P \vee P \\ & | & \neg P \end{array}$$

Resolving Ambiquity

	expression	how to read it	how not to read it
Algebra	$3 + 4 \times 5$	$3 + (4 \times 5) = 23$	$(3 + 4) \times 5 = 35$
	$3 \times 4 + 5$	$(3 \times 4) + 5$	$3 \times (4 + 5)$

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Boolean Algebra	interpretation I	interpretation II
$\text{true} \vee \text{false} \wedge \text{false}$	$\text{true} \vee (\text{false} \wedge \text{false})$ evaluates to true	$(\text{true} \vee \text{false}) \wedge \text{false}$ evaluates to false

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  Parentheses must be used whenever ambiguity would result from their omission

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Negation binds tightly: $\neg P \wedge Q$ is not equivalent to $\neg(P \wedge Q)$.

Ambiguity in English

Consider the phrase

you can have soup or salad and pasta.

If the intended meaning is "soup or (salad and pasta)":

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you can have soup or salad and pasta.

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you can have soup, or salad and pasta

If the intended meaning is "(soup or salad) and pasta":

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you can have soup, or salad and pasta

If the intended meaning is "(soup or salad) and pasta":

you can have soup or salad, and pasta

you can have soup or salad and also pasta

you can have pasta and either soup or salad

The Game in Tarski's World

- Given sentence $P = \texttt{Cube}(\texttt{c}) \vee \texttt{Cube}(\texttt{d})$.
- Given world where c is a cube but d is not.

We	Opponent
P is false in this world	
	So c is not a cube?
Eh. . . I admit defeat	

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OK, P is true in this world	
	Because c is a cube or because d is?
Because d is a cube	
	You lost but could have won

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Because d is a cube	
	You lost but could have won
OK, because c is a cube	
	You won (finally!)

More about the Game

- Given sentence $P = \texttt{Cube(a)} \vee \neg\texttt{Cube(a)}$.

We	Opponent
P is true in this world	
	Because a is a cube or because a is not a cube?
Eh. . . I don't know but P will always be true!	
	Please answer my question!

- Who won the game???