

Proving the Validity of an Argument

Torben Amtoft
Kansas State University

Introduction

Proving Validity: Examples

The Notion of Proof Rules

Proving Validity: More Examples

Fitch

Reasoning about Identity

Reminder

In Fitch format, an **argument** takes the form

$$\begin{array}{|l}
 P_1 \\
 \dots \\
 P_n \\
 \hline
 Q
 \end{array}
 \quad
 \begin{array}{l}
 \text{premises} \\
 \\
 \text{conclusion}
 \end{array}$$

- ▶ Such an argument is **valid** if conclusion Q is true whenever premises $P_1 \dots P_n$ are.
- ▶ A valid argument is **sound** if the premises are true.

Proving and Disproving Validity

- ▶ To show that an argument is **not** valid, use **counterexamples**: a world where the premises are true, but conclusion false.
- ▶ How to show that an argument **is** valid?
(topic of Section 2.2-2.4)
 - ▶ **naive** approach: consider all worlds where premise is true, and show that also conclusion is true.
 - ▶ **feasible** approach: construct a **proof**

A Valid Argument

RightOf(b,c)

LeftOf(d,e)

b = d

LeftOf(c,e) why?

A Valid Argument

$$\text{RightOf}(b, c)$$

$$\text{LeftOf}(d, e)$$

$$b = d$$

$$\text{LeftOf}(c, e) \quad \text{why?}$$

Informal reasoning: (p.52)

We are told that b is to the right of c . So c must be to the left of b , since *right of* and *left of* are inverses of one another. And since $b = d$, c is left of d , by the indiscernibility of identicals. But we are also told that d is left of e , and consequently c is to the left of e , by the transitivity of *left of*. This is our desired conclusion.

A Formal Proof

We establish a series of intermediate results:

- | | |
|---------------------------|---|
| 1. $\text{RightOf}(b, c)$ | |
| 2. $\text{LeftOf}(d, e)$ | |
| 3. $b = d$ | |
| 4. $\text{LeftOf}(c, b)$ | from 1, since LeftOf and RightOf are inverses |
| 5. $\text{LeftOf}(c, d)$ | from 4 and 3, using identity elimination |
| 6. $\text{LeftOf}(c, e)$ | from 5 and 2, since LeftOf is transitive |

Identity Elimination (= Elim)

- ▶ If b is a cube and b equals c , then also c is a cube
- ▶ If John is happy and John is the father of Max, then the father of Max is happy

In general (cf. p. 56)

$$\triangleright \left| \begin{array}{l} P(t) \\ \dots \\ t = u \\ \dots \\ P(u) \end{array} \right.$$

A Valid Arithmetic Argument

$$x > 2$$

$$x \cdot x > x + x \quad \text{why?}$$

A Valid Arithmetic Argument

- | | | |
|----|-------------------------|---|
| 1. | $x > 2$ | |
| 2. | $2 > 0$ | fact |
| 3. | $x > 0$ | from 1 and 2, since $>$ is transitive |
| 4. | $x \cdot x > 2 \cdot x$ | from 1 and 3, using law of arithmetic:
multiplying by positive number preserves inequalities |
| 5. | $2 \cdot x = x + x$ | fact |
| 6. | $x \cdot x > x + x$ | from 4 and 5, using $=$ Elim |

System F, vs. Fitch

	system F (p.54)	Fitch (p.58)
what is?	formal system	software package
application domain	general purpose	tuned for Tarski's world
Proof rules:		
general	= Elim, = Intro \wedge Elim, \wedge Intro etc.	= Elim, = Intro \wedge Elim, \wedge Intro etc.
shortcuts	can be added	Taut Con, FO Con
specific	can be added	Ana Con (encodes block world laws)

Proof in Fitch

Converting previous proof into Fitch:

- | | |
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Converting previous proof into Fitch:

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2. $\text{LeftOf}(d, e)$
3. $b = d$

4. $\text{LeftOf}(c, b)$ **Ana Con:** 1
5. $\text{LeftOf}(c, d)$ from 4 and 3, using identity elimination
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| 3. $b = d$ | |
| 4. $\text{LeftOf}(c, b)$ | Ana Con: 1 |
| 5. $\text{LeftOf}(c, d)$ | = Elim: 4, 3 |
| 6. $\text{LeftOf}(c, e)$ | from 5 and 2, since LeftOf is transitive |

Proof in Fitch

Converting previous proof into Fitch:

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| 2. LeftOf(d,e) | |
| 3. $b = d$ | |
| 4. LeftOf(c,b) | Ana Con: 1 |
| 5. LeftOf(c,d) | = Elim: 4, 3 |
| 6. LeftOf(c,e) | Ana Con: 5, 2 |

Limitations on Fitch

- ▶ Fitch is tuned for **Tarski's World**,
with **Ana Con** modeling the laws of the block world
- ▶ Fitch is **not** tuned for arithmetic.
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- ▶ System F is not tuned for Tarski's world, and therefore has no special proof rules for the predicates there, like `LeftOf`
- ▶ But since the identity relation “=” is used in all domains of discourse, system F has special rules for that predicate

Identity Introduction (= Intro)

Rule described p.55 is amazingly simple:

$$\triangleright \mid t = t$$

This says that the identity relation is **reflexive**.

Properties of Identity

Symmetry

$$\begin{array}{|l} \hline a = b \\ \hline b = a \end{array}$$

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Transitivity

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$$\begin{array}{l|l} 1. a = b & \\ \hline 2. a = a & = \text{Intro} \\ 3. b = a & = \text{Elim: 2, 1} \end{array}$$

Transitivity

$$\begin{array}{l|l} a = b & \\ b = c & \\ \hline a = c & \end{array}$$

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Transitivity

$$\begin{array}{l} 1. a = b \\ 2. b = c \\ \hline 3. a = c \quad = \text{Elim: 1, 2} \end{array}$$