BCU Mathematics Contest 1999

Edited by Finbarr Holland & Pascal Hitzler

March 13, 1999

BCU Mathematics Contest, MARCH 13, 1999

TEAM CONTEST PAPER

Answer *all* questions.

Each question is worth 10 points.

Contestants have 45 minutes to do the work.

Teams are permitted to submit *at most* one solution per question. It's therefore up to each team to select a solution that, in their opinion, will attract the maximum score.

1. In how many ways can the number of days in a non-leap year be written in the form $ab + a + b$, where a, b are positive integers?

   [10 points]

2. Let
$$a_n = \frac{2}{n^2 - 1}, \; n = 2, 3, \ldots$$
   Determine real numbers A, B so that
$$a_n = \frac{A}{n-1} + \frac{B}{n+1}, \; n = 2, 3, \ldots$$
   Define s_k for $k = 2, 3, \ldots$ by
$$s_k = a_2 + a_3 + \cdots + a_k = \sum_{n=2}^{k} a_n, \; k = 2, 3, \ldots$$

Show that

$$\frac{4}{25} < s_{100} - s_{10} < \frac{1}{5}.$$

[10 points]

3. Sketch the graph of the function

$$y = x^3 - 6x^2 + 12x - 7.$$

For what range of x is it concave? convex? What are its points of inflection?

[10 points]

4. Determine the area of the bounded region enclosed by the set of points in the xy-plane that satisfy the equation

$$xy + y^2 - yx^2 - x^3 = 0.$$

[10 points]

BCU Mathematics Contest, MARCH 13, 1999

INDIVIDUAL CONTEST PAPER

Answer *all* questions.

Questions 1 and 3 are worth 10 points; Question 2 is worth 5 points.

Contestants have 45 minutes to do the work.

1. At each vertex v of a wooden cube of dimension $2a \times 2a \times 2a$ a tetrahedron is removed by cutting along a plane which intersects the midpoints of all the edges adjacent to v.

   Calculate the volume and surface area of the remaining solid.

   [10 points]

2. Suppose a, b, c are real numbers with $a \neq 0$, $b > 0$ and $b^2 > 4ac$. Let β be the largest root of the equation

   $$ax^2 + bx + c = 0.$$

   Determine

   $$\lim_{a \to 0} \beta.$$

   [5 points]

3. Points X, Y are chosen on a semicricle erected on a diameter AB of length d units so that the angle $\angle BAY$ is twice the angle $\angle YAX$. Let C denote the point of intersection of the chords AY and BX, and let E be the point on AY (extended) so that Y is the midpoint of the segment CE.

   Determine the length of AE.

   [10 points]

BCU Mathematics Contest, MARCH 13, 1999

SPEED CONTEST PAPER FOR TEAMS

Questions 2 and 7 are worth 6 points; Questions 4,5 and 6 are worth 5 points, and Questions 1 and 3 are worth 2 points.

Contestants have 45 minutes to attempt a maximum of 7 questions.
Questions will be distributed one at a time.

Only the team who hands up the correct solution in the shortest time earns the marks assigned to a particular question; all other teams earn no marks on that question. As soon as a correct answer to a question is submitted to the invigilators the next question will be distributed to all the teams. The team score for this contest will be the sum of its scores on the questions handed out in the time period set for the contest.

1. Initially, 55% of the participants at a summer school were girls. Then 5 more girls joined which caused the percentage of girls to rise to 60%. How many people participated in the summer school initially, and how many of these were girls?

   [2 points]

2. Determine the points on the circle

$$x^2 + y^2 = 1$$

where the function

$$f(x, y) = 2x^2 + 3y^2$$

attains its maximum and minimum values.

[6 points]

3. Suppose a, b are natural numbers such that $b < a$. Show that

$$\frac{b - a^2}{2a} \leq 1.$$

[2 points]

4. Determine the radii of the inscribed and circumscribed circles of an equilateral triangle the length of whose sides is a.

[5 points]

5. A sequence of polynomials $p_1, p_2, \ldots, p_n, \ldots$ satisfies the recursion formula

$$p_{n+2}(x) = p_{n+1}(x) + 2xp_n(x), \ n = 1, 2, \ldots, \ x \in \mathcal{R}.$$

If

$$p_1(x) = 1, \ p_2(x) = x + 3,$$

determine $p_5(0)$ and $p_5(-1)$.

[5 points]

6. What are the last two digits in the expansion of the number 7^{76} in base 10?

[5 points]

7. On two different days each member of a group of 45 students takes exactly two or three of the courses Spanish, Mathematics, History and Computer Science. If 34 students take History or Spanish, and 27 students take three courses, at least how many students take Mathematics or Computer Science?

[6 points]