Characterizing Φ-accessible Programs

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Abstract

We characterize programs with a total Fitting semantics, recovering from some mistakes in [Hit01].

Notation and terminology is that of [Hit01].

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Roland Heinze found the problem with the original definition, see Remark 0.4.

0.1 Theorem Let P be a normal logic program. The following conditions are equivalent.

1) P has a total Fitting semantics.

2) There exists a model I and a level mapping l such that I is a supported model of P and each $A \in B_P$ satisfies either (i) or (ii).

   (i) There exists a clause $A \leftarrow L_1, ..., L_n$ in ground(P) with head A such that $I \models L_1 \wedge \cdots \wedge L_n$, and $l(A) > l(L_i)$ for all i.

   (ii) For each clause $A \leftarrow L_1, ..., L_n$ in ground(P) with head A there exists i such that $I \not\models L_i$ and $l(A) > l(L_i)$.

3) There exists a model I and a level mapping l such that I is a model of P and each $A \in B_P$ satisfies either (i) from 2) above or (iii).

   (iii) $I \not\models A$ and for each clause $A \leftarrow L_1, ..., L_n$ in ground(P) with head A there exists i such that $I \not\models L_i$ and $l(A) > l(L_i)$.

Furthermore, I is the unique supported model of P.

Proof: 3) $\Rightarrow$ 2). Suppose 3) holds. We show that I is supported. So let $A \in I$. Then (iii) can not hold. So (i) holds by condition 3). Hence there exists a clause in ground(P) with head A whose body is true under I. So I is supported. Furthermore, if $A \in I$, then (i) holds. If $A \notin I$, then (i) can not hold since I is a model for P. Hence (iii) holds by condition 3). Consequently, (ii) holds in this case.

2) $\Rightarrow$ 3). Suppose A does not satisfy (i). Then it satisfies (ii). Since I is supported, we must have $A \notin I$ which shows (iii).

1) $\Leftrightarrow$ 2). This is exactly the proof of Theorem 6.5.3 in [Hit01]. We replicate it here.

1) $\Rightarrow$ 2). Suppose P satisfies condition 1). For each $A \in B_P$, let $l_P(A)$ denote the least ordinal β such that A is not undefined in $\Phi_P\uparrow(\beta+1)$. Let α be its closure ordinal wrt. Φ_P and let $M_P = \Phi_P\uparrow\alpha^+$ be its unique supported (two-valued) model. We distinguish two cases (a) and (b).
(a) Let $A \in M_P$ and $l_P(A) = \beta$. By definition of l_P and Φ_P there exists a clause $A \leftarrow L_1, \dots, L_n$ in ground(P) such that the $L_1, \dots, L_n$ are true in $\Phi\uparrow\beta$, and hence are also true in M_P. Again by definition of l_P we obtain $l_P(A) > l_P(L_i)$ for all i.
(b) Let $A \notin M_P$ and $l_P(A) = \beta$. By definition of l_P and Φ_P we obtain that for any clause $A \leftarrow L_1, \dots, L_n$ in ground(P) we must have that $L_1 \wedge \dots \wedge L_n$ is false in $\Phi_P\uparrow\beta$. So there must be some i such that L_i is false in $\Phi_P\uparrow\beta$ and $l(L_i) < \beta$ by definition of l_P, and hence $l_P(A) > l_P(L_i)$.
Thus, P satisfies condition 2) with $I = M_P$ and $l = l_P$.

2) $\Rightarrow$ 1) Assume P satisfies condition 2). We show by induction on β that any $A \in B_P$ with $l(A) = \beta$ is not undefined in $\Phi_P\uparrow(\beta+1)$ and, furthermore, that I and $\Phi_P\uparrow(\beta+1)$ agree on A.
If $l(A) = 0$, then A must be the head of a unit clause or does not appear in any head. In the first case, A is true in $\Phi_P\uparrow 1$, and in the second case, A is false in $\Phi_P\uparrow 1$. Note that in the first case A is also true in I since condition (i) applies and I is a model of P. Also, in the second case, A is also false in I since condition (ii) applies and I is supported.
Now let $l(A) = \beta$. If there is no clause in ground(P) with head A, then A is false in $\Phi_P\uparrow 1 \leq \Phi_P\uparrow(\beta+1)$ and also false in I since condition (ii) applies and I is supported. So assume there is a clause in ground(P) with head A. By hypothesis, either condition (i) or condition (ii) applies.
If condition (i) applies, then there is a clause $A \leftarrow L_1, \dots, L_n$ in ground(P) such that $l(L_1), \dots, l(L_n) < l(A)$ and therefore, by the induction hypothesis, the $L_1, \dots, L_n$ are not undefined in $\Phi_P\uparrow\beta$ and I agrees with $\Phi_P\uparrow\beta$ on them. Now, since I is a model of P and $I \models L_1, \dots, L_n$, we obtain that A is true in I and by definition of Φ_P also in $\Phi_P\uparrow(\beta+1)$.
If condition (ii) applies, then for each clause $A \leftarrow L_1, \dots, L_n$ in ground(P) there is some i such that $l(A) > l(L_i)$ and L_i is false in I. Hence we obtain that L_i is false in both $\Phi_P\uparrow\beta$ and I by the induction hypothesis and it follows that A is false in $\Phi_P\uparrow(\beta+1)$ by definition of Φ_P and also false in I since I is supported.

By 2), I is supported. By 1), P has a unique supported model. Hence I is the unique supported model of P. ■

The following definition replaces the respective part of [Hit01, Definition 5.0.2].

0.2 Definition A normal logic program is called Φ-*accessible* if it satisfies one of the equivalent conditions from Theorem 0.1.

0.3 Remark The following condition is *not* equivalent to Φ-accessibility: There exists a model I and a level mapping l such that I is a model of P whose restriction to the predicate symbols in Neg_P^* is a supported model of P^-, and each $A \in B_P$ satisfies either (i) or (ii) from 2) above.

Proof: The following program is a counterexample:

$$p \leftarrow q$$
$$q \leftarrow r$$
$$q \leftarrow p$$

It satisfies the above conditions for the model $I = \{p, q, r\}$ and the level mapping $l(p) = 2 > l(q) = 1 > l(r) = 0$. The program has no total Fitting semantics. ■

0.4 Remark (Heinze) The following condition is *not* equivalent to Φ-accessibility: There exists a model I and a level mapping l such that I is a model of P and each $A \in B_P$ satisfies either (i) from 2) above or (iv).

(iv) For each clause $A \leftarrow L_1, ..., L_n$ in $ground(P)$ with head A there exists i such that $I \not\models L_i$, $I \not\models A$ and $l(A) > l(L_i)$.

Proof: The following program is a counterexample.

$$p \leftarrow \neg p, \neg q$$

It satisfies the above conditions for the model $I = \{q\}$ and the level mapping $l(p) = 1 > l(q) = 0$. The program has no total Fitting semantics.

Note that the program from Remark 0.3 also serves as a counterexample. ■

We note that the proof of [Hit01, Proposition 5.5.3] can be carried over using that I is supported. We repeat it for convenience.

0.5 Proposition Let P be Φ-accessible. Then T_P is strictly contracting with respect to ϱ.

Proof: Let $J, K \in I_P$ and assume that $\varrho(J, K) = 2^{-\alpha}$. Then J, K, I agree on all ground atoms of level less than α. We show that $T_P(J)$ and I agree on all ground atoms of level less than or equal to α. A similar argument shows that $T_P(K)$ and I agree on all ground atoms of level less than or equal to α, and this suffices.

Let $A \in T_P(J)$ with $l(A) \leq \alpha$. Then there must be a clause $A \leftarrow L_1, \ldots, L_n$ in ground(P) such that $J \models L_1 \wedge \cdots \wedge L_n$. Since I and J agree on all ground atoms of level less than α, condition (ii) from Theorem 0.1 2) cannot hold, because if $I \not\models L_i$ with $l(A) > l(L_i)$, then $J \not\models L_i$ and consequently $J \not\models L_1 \wedge \cdots \wedge L_n$, which is a contradiction. Therefore, condition (i) of Theorem 0.1 2) holds and so $A \in T_P(I)$. Since I is supported and $T_P(I) = I$ we conclude $A \in I$.

Conversely, suppose that $A \in I$. Since $I = T_P(I)$, there must be a clause $A \leftarrow L_1, \dots, L_n$ in ground(P) such that $I \models L_1 \wedge \cdots \wedge L_n$. Thus, condition (i) of Theorem 0.1 2) must hold, and so we can assume that $A \leftarrow L_1, \dots, L_n$ also satisfies $l(A) > l(L_i)$ for $i = 1, \dots, n$. Since I and J agree on all ground atoms of level less than α, we have $J \models L_1 \wedge \cdots \wedge L_n$ and hence $A \in T_P(J)$ as required. ■

Finally, we note that the proof of [Hit01, Theorem 8.2.2] is unaffected.

References

[Hit01] P. Hitzler. *Generalized Metrics and Topology in Logic Programming Semantics.* PhD thesis, Department of Mathematics, National University of Ireland, University College Cork, 2001.