Generalized Metric Spaces in Logic Programming Semantics

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Logic Programs: Fixpoint Semantics

A logic program P is a finite set of clauses

$$\forall(A \leftarrow L_1 \wedge \cdots \wedge L_n)$$

from first order logic usually written as

$$A \leftarrow L_1 \wedge \cdots \wedge L_n,$$

where A an atom, L_i a literal, $n \geq 0$.
B_P: Herbrand base.
$I_P = 2^{B_P}$: set of all Herbrand interpretations.
ground(P): set of all ground clauses of P.

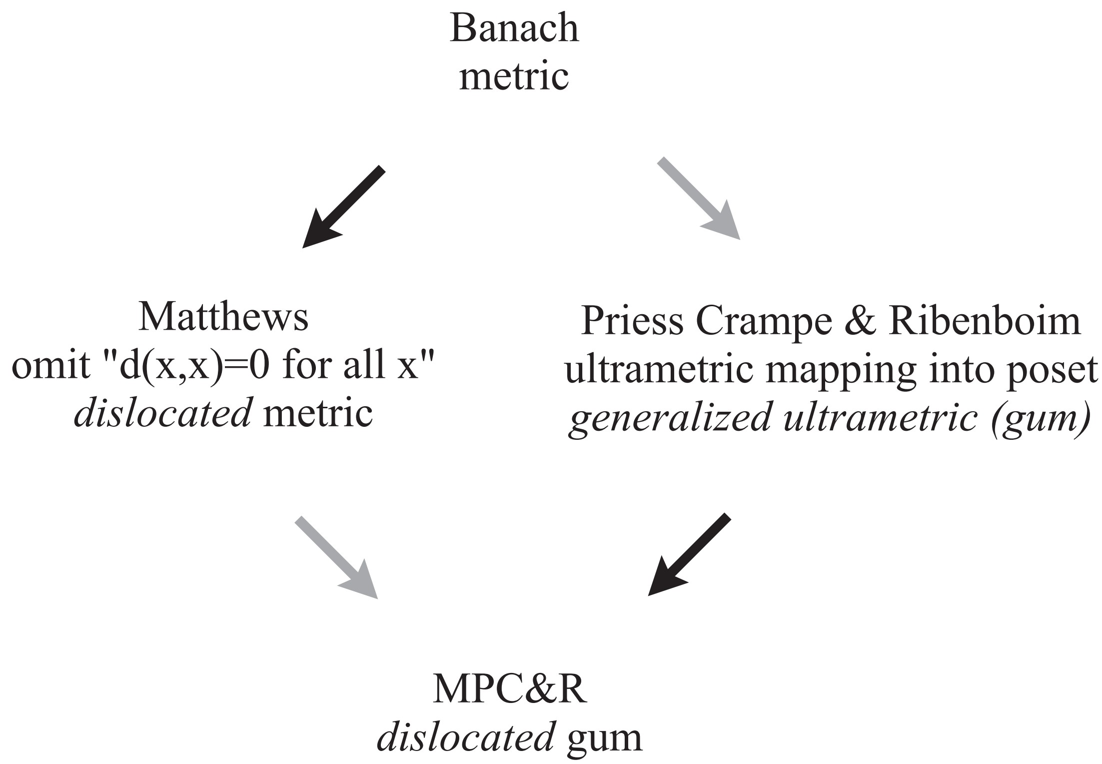
Denotational semantics is given by models with additional properties.
We focus on the *supported model semantics.*

Define (nonmonotonic) operator $T_P : I_P \to I_P$ by
$T_P(I)$ is set of all $A \in B_P$
for which there is a clause $A \leftarrow L_1 \wedge \cdots \wedge L_n$
in ground(P) s.t. $I \models L_1 \wedge \cdots \wedge L_n$.

I is a *supported model* iff $T_P(I) = I$.

We seek ways of finding fixed points of T_P.

Generalized Metric Fixpoint Theorems



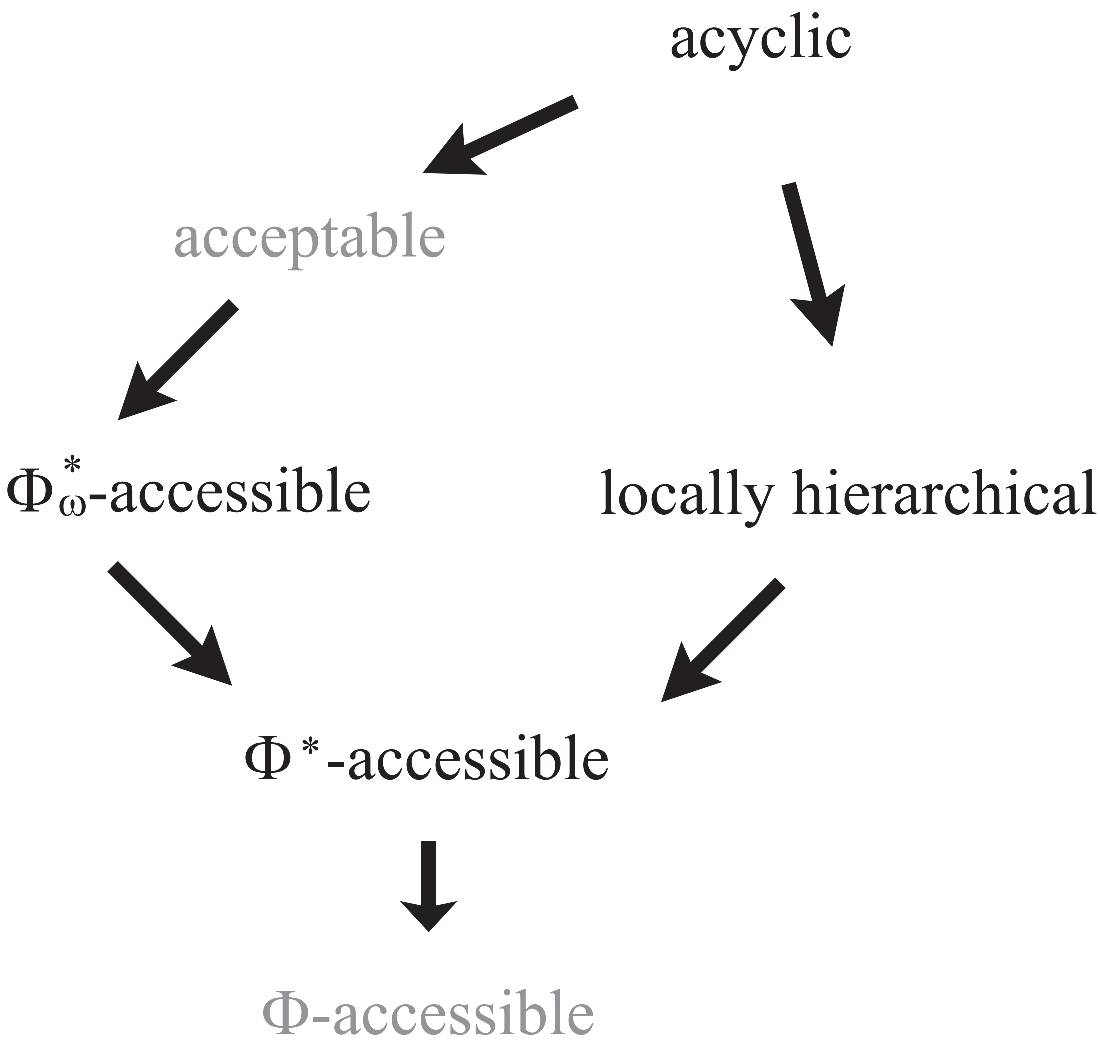
Theorems, if applicable, yield unique fixed points.

Programs which can be analysed with them will have unique supported models
(i.e. are *uniquely determined*).

∗ Matthews 1985

∗ Priess-Crampe & Ribenboim 2000

Classes of Uniquely Determined Programs



A *level mapping* is a function $l : B_P \rightarrow \gamma_P$, where γ_P is a (countable) ordinal.

P is *locally hierarchical*(*lh*) if
for each $A \leftarrow L_1, \ldots, L_n$ in ground(P),
$l(A) > l(L_i)$ for all i.

P is *acyclic* if it is lh and $\gamma_P = \omega$.

$*$ Cavedon 1989, Apt&Pedreschi 1993, H&S 1999

Basic Construction

P logic program.

l level mapping for P.

For $J, K \in I_P$ define
$d(J,K) = 0$ if $J = K$ and
$d(J,K) = 2^{-\alpha}$
where J, K disagree on $A \in B_P$ with $l(A) = \alpha$
and agree on all atoms of level less than α.
$(2^{-\alpha} < 2^{-\beta}$ iff $\beta < \alpha)$

If P acyclic:

- (I_P, d) is complete ultrametric space.
- T_P is a contraction relative to d.
- T_P has unique fixed point.
- P has unique supported model M.
- $T_P^n(K) \to M$ in the Cantor topology on I_P (for all $K \in I_P$).

Generalized Ultrametrics

P locally hierarchical:

• (I_P, d) *generalized ultrametric* (*gum*), i.e.

∘ $d : X \times X \to \Gamma$ $(X = I_P)$, Γ poset, $\min \Gamma = 0$

∘ $d(x, y) = 0$ iff $x = y$ (for all x, y)

∘ $d(x, y) = d(y, x)$ (for all x, y)

∘ $d(x, y) \leq \gamma$ and $d(y, z) \leq \gamma \Rightarrow d(x, z) \leq \gamma$ (for all x, y, z, γ)

• (I_P, d) *spherically complete* i.e.

$\bigcap \mathcal{C} \neq \emptyset$ for each chain $\mathcal{C}$ of (nonempty) *balls* $(B_\gamma(y) = \{x \mid d(x, y) \leq \gamma\})$.

• T_P *strictly contracting* i.e.

$d(T_P(x), T_P(y)) < d(x, y)$ for all $x \neq y$.

• T_P has unique fixed point.

PC&R Theorem: (X, d) sph. comp. gum, f str. contr., then f has a unique fixed point.

• P has unique supported model M.

• M can be obtained as the limit of a transfinite iterative process involving T_P and the Cantor topology on I_P.

Domains as Gums

D algebraically complete cpo (e.g. I_P).
γ countable ordinal, $\Gamma_\gamma = \{2^{-\alpha} \mid \alpha < \gamma\}$.

$r : D_C \to \gamma + 1$ *rank function.*
$d_r : D \times D \to \Gamma_{\gamma+1}$ defined by
$d_r(x, y) = \inf\{2^{-\alpha} \mid$
$(c \sqsubseteq x \text{ iff } c \sqsubseteq y)$ for all $c \in D_C$ with $r(c) < \alpha\}$.

(D, d_r) is a spherically complete gum.

Proof uses the following observations:

- $x \in B_{2^\beta}(y) \Rightarrow \{c \in \text{approx}(x) \mid r(c) < \beta\} = \{c \in \text{approx}(y) \mid r(c) < \beta\}$
- $B_\beta = \sup\{c \in \text{approx}(y) \mid r(c) < \beta\}$ exists
- $B_\beta \in B_{2^\beta}(y)$
- $B_{2^\alpha}(x) \subseteq B_{2^\beta}(y) \Rightarrow B_\beta \sqsubseteq B_\alpha$

* cf. Smyth 1989/91
* generalizes earlier result Seda & Hitzler 1997
* PC&R Theorem is more general than applied
* bottom element of D not needed

Dislocated Metrics

(X, ϱ) *dislocated metric space* (*d-metric*)
(Matthews 1985: *metric domain*):

ϱ satisfies all conditions of a metric *except*

∘ $\varrho(x, x) = 0$ for all $x \in X$.

Remaining notions as in the metric case.

Matthews: (X, ϱ) complete d-metric space,
$f : X \to X$ contraction.
Then f has a unique fixed point.

P is Φ^*-*accessible* if
∘ I model for P
∘ I supported model of P^- (negative part of P)
∘ l level mapping for P s.t.
for all $A \leftarrow L_1, \dots, L_n$ in ground(P) either
$I \models L_1 \wedge \dots \wedge L_n$ and $l(A) > l(L_i)$ for all i or
exists i s.t. $I \not\models L_i$ and $l(A) > l(L_i)$.

P is Φ^*_ω-*accessible* if it is Φ^*-accessible
and $\gamma_P = \omega$.

$*$ cf. Apt & Pedreschi 1993: acceptable programs

Generalized Fitting Construction

Neg_P^*: predicates occurring negatively in P
and all predicates on which they depend.

P^-: all ground clauses with head from Neg_P^*.

$K \in I_P$, then K' is K restricted to predicates not in N.

Definitions: $I \in I_P$ and level map l fixed.

- ∘ $f(K) = 0$ if $K \subseteq I$.
- ∘ $f(K) = 2^{-\alpha}$ with α least s.t. exists $A \in K \setminus I$ with $l(A) = \alpha$.
- ∘ $u(K) = \max\{f(K'), d(K', I)\}$.
- ∘ $\varrho(J, K) = \max\{d(J, K), u(J), u(K)\}$

P is Φ_ω^*-accessible:

- (I_P, ϱ) complete d-metric.
- T_P contraction.
- T_P has unique fixed point.
- P has unique supported model M.
- $T_P^n(K) \to M$ in the Cantor topology on I_P (for all $K \in I_P$).

∗ cf. Fitting 1994

Dislocated GUMs

(X, ϱ) *dislocated gum* (*d-gum*):

ϱ satisfies all conditions of a gum *except*

∘ $\varrho(x, x) = 0$ for all $x \in X$.

Remaining notions as in the gum case.

(X, ϱ) spherically complete d-gum,
$f : X \to X$ strictly contracting.
Then f has a unique fixed point.

P Φ^*-accessible:

- (I_P, ϱ) spherically complete d-gum.
- T_P strictly contracting.
- T_P has unique fixed point.
- P has unique supported model M.
- M can be obtained as the limit of a transfinite iterative process involving T_P and the Cantor topology on I_P.

Another Application

P is Φ-*accessible* if I model for P and l level map s.t. each $A \in B_P$ satisfies either (i) or (ii).

(i) Exists $A \leftarrow L_1, \dots, L_n$ in ground(P) s.t. $I \models L_1 \wedge \cdots \wedge L_n$ and $l(A) > l(L_i)$ for all i.

(ii) For each $A \leftarrow L_1, \dots, L_n$ in ground(P) exists i with $I \not\models L_i$, $I \not\models A$, $l(A) > l(L_i)$.

$*$ P is Φ-accessible iff P has total model under Fitting-semantics (Fitting 1985).

Define $\delta(J, K) = \max\{d(J, I), d(K, I)\}$.

- (I_P, δ) spherically complete d-gum.
- T_P strictly contracting.
- M can be obtained as the limit of a transfinite iterative process involving T_P and the Cantor topology on I_P.

Discussion

- Understanding nonmonotonic reasoning.

  Rounds & Zhang 199x

- Exploring the "space" of all logic programs.

- Extensions to uncertain reasoning?

  van Emden 1986
  Mateis 1999

- Connections to topological dynamics.

  Seda & Hitzler 1997/8

- Relationships to artificial neural networks.

  Hölldobler et al. 199x
  Hitzler & Seda 2000

Appendix: Atomic Topology Q on I_P

B_P countable
then Q homeomorphic to Cantor set.

Equivalent characterizations:

- Product toplogy on 2^{B_P}
  where $2 = \{0, 1\}$ carries discrete topology.
- Subbase $\{\mathcal{G}(L) \mid L \text{ literal}\}$,
  $\mathcal{G}(L) = \{I \in I_P \mid I \models L\}$.
- $I_n \to I$ if
  each $A \in I$ is eventually in I_n and
  each $A \notin I$ is eventually not in I_n.

For the transfinite iterative processes above:

For limit ordinal α, set I_α to be set of all $A \in B_P$ which are eventually in $(I_\beta)_{\beta<\alpha}$.

For successor ordinal α, set $I_\alpha = T_P(I_{\alpha-1})$.

Transfinite sequence obtained converges in Q.

$*$ Batarekh & Subrahmanian 1989:
Query Topology

Appendix: General Version of MPC&R

(X, ϱ) spherically complete d-gum.
$f : X \to X$ *non-expanding*
$(d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X)$
and f *strictly contracting on orbits*
$(d(f^2(x), f(x)) < d(f(x), x)$
for all $x \in X$ with $x \neq f(x))$.
Then f has fixed point.
f strictly contracting then fixed point is unique.

- Also generalizes theorem by Khamsi, Kreinovich & Misane 1993.
- Constructive proof for applied special case possible.

Level mappings to rank functions:

$I \in I_P$ finite set (i.e. compact element of I_P).

$r(I) = \max\{l(A) \mid A \in I\}$.

Appendix: Metrics and D-metrics

(X, d) complete ultrametric.
$u : X \to \mathbb{R}_0^+$ continuous.

Then $\varrho(x, y) = \max\{d(x, y), u(x), u(y)\}$
is complete d-ultrametric.

(X, ϱ) complete d-metric.

$d(x, y) = 0$ if $x = y$,
$d(x, y) = \varrho(x, y)$ otherwise.

Then d complete metric.

If f contraction in ϱ
then f contraction in d.

• Can prove the theorem of Matthews from Banach theorem.

• Anologous results hold for d-gums/gums.