Generalized Metric Spaces in Logic Programming Semantics

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Logic Programs: Fixpoint Semantics

A logic program $P$ is a finite set of clauses

$$\forall (A \leftarrow L_1 \land \cdots \land L_n)$$

from first order logic usually written as

$$A \leftarrow L_1 \land \cdots \land L_n,$$

where $A$ an atom, $L_i$ a literal, $n \geq 0$.

$B_P$: Herbrand base.

$I_P = 2^{B_P}$: set of all Herbrand interpretations.

$\text{ground}(P)$: set of all ground clauses of $P$.

Denotational semantics is given by models with additional properties.

We focus on the supported model semantics.

Define (nonmonotonic) operator $T_P : I_P \rightarrow I_P$ by $T_P(I)$ is set of all $A \in B_P$

for which there is a clause $A \leftarrow L_1 \land \cdots \land L_n$

in $\text{ground}(P)$ s.t. $I \models L_1 \land \cdots \land L_n$.

$I$ is a supported model iff $T_P(I) = I$.

We seek ways of finding fixed points of $T_P$. 
Theorems, if applicable, yield unique fixed points.

Programs which can be analysed with them will have unique supported models (i.e. are uniquely determined).

* Matthews 1985
* Priess-Crampe & Ribenboim 2000
A level mapping is a function $l : B_P \to \gamma_P$, where $\gamma_P$ is a (countable) ordinal.

$P$ is locally hierarchical (lh) if
for each $A \leftarrow L_1, \ldots, L_n$ in $\text{ground}(P)$,
$l(A) > l(L_i)$ for all $i$.

$P$ is acyclic if it is lh and $\gamma_P = \omega$.

Basic Construction

$P$ logic program.

$l$ level mapping for $P$.

For $J, K \in I_P$ define

$d(J, K) = 0$ if $J = K$ and

$d(J, K) = 2^{-\alpha}$

where $J, K$ disagree on $A \in B_P$ with $l(A) = \alpha$

and agree on all atoms of level less than $\alpha$.

$(2^{-\alpha} < 2^{-\beta} \text{ iff } \beta < \alpha)$

If $P$ acyclic:

- $(I_P, d)$ is complete ultrametric space.
- $T_P$ is a contraction relative to $d$.
- $T_P$ has unique fixed point.
- $P$ has unique supported model $M$.
- $T^n_P(K) \rightarrow M$ in the Cantor topology on $I_P$
  (for all $K \in I_P$).
Generalized Ultrametrics

$P$ locally hierarchical:

- $(I_P, d)$ generalized ultrametric (gum), i.e.
  - $d : X \times X \to \Gamma$ ($X = I_P$), $\Gamma$ poset, $\min \Gamma = 0$
  - $d(x, y) = 0$ iff $x = y$ (for all $x, y$)
  - $d(x, y) = d(y, x)$ (for all $x, y$)
  - $d(x, y) \leq \gamma$ and $d(y, z) \leq \gamma \Rightarrow d(x, z) \leq \gamma$
    (for all $x, y, z, \gamma$)

- $(I_P, d)$ spherically complete i.e.
  \[ \bigcap \mathcal{C} \neq \emptyset \text{ for each chain } \mathcal{C} \text{ of (nonempty) balls } \]
  \[ (B_\gamma(y) = \{ x \mid d(x, y) \leq \gamma \}). \]

- $T_P$ strictly contracting i.e.
  \[ d(T_P(x), T_P(y)) < d(x, y) \text{ for all } x \neq y. \]

- $T_P$ has unique fixed point.

**PC&R Theorem:** $(X, d)$ sph. comp. gum, $f$ str. contr., then $f$ has a unique fixed point.

- $P$ has unique supported model $M$.
- $M$ can be obtained as the limit of a transfinite iterative process involving $T_P$ and the Cantor topology on $I_P$. 

Domains as Gums

$D$ algebraically complete cpo (e.g. $I_P$).
\[ \gamma \text{ countable ordinal, } \Gamma_\gamma = \{2^{-\alpha} \mid \alpha < \gamma\}. \]

$r : D_C \rightarrow \gamma + 1$ rank function.

$d_r : D \times D \rightarrow \Gamma_{\gamma+1}$ defined by
\[ d_r(x, y) = \inf\{2^{-\alpha} \mid (c \sqsubseteq x \text{ iff } c \sqsubseteq y) \text{ for all } c \in D_C \text{ with } r(c) < \alpha\}. \]

$(D, d_r)$ is a spherically complete gum.

Proof uses the following observations:

- $x \in B_{2^\beta}(y) \Rightarrow \{c \in \text{approx}(x) \mid r(c) < \beta\}$
  \[ = \{c \in \text{approx}(y) \mid r(c) < \beta\} \]
- $B_\beta = \sup\{c \in \text{approx}(y) \mid r(c) < \beta\}$ exists
- $B_\beta \in B_{2^\beta}(y)$
- $B_{2^\alpha}(x) \subseteq B_{2^\beta}(y) \Rightarrow B_\beta \sqsubseteq B_\alpha$

* cf. Smyth 1989/91
* generalizes earlier result Seda & Hitzler 1997
* PC&R Theorem is more general than applied
* bottom element of $D$ not needed
**Dislocated Metrics**

$(X, \varrho)$ *dislocated metric space* (d-metric)  
(Matthews 1985: *metric domain*):

- $\varrho$ satisfies all conditions of a metric *except*
  - $\varrho(x, x) = 0$ for all $x \in X$.

Remaining notions as in the metric case.

Matthews: $(X, \varrho)$ complete d-metric space, $f : X \to X$ contraction.  
Then $f$ has a unique fixed point.

$d$ is $\Phi^*$-*accessible* if
- $I$ model for $P$
- $I$ supported model of $P^{-}$ (negative part of $P$)
- $l$ level mapping for $P$ s.t.
  - for all $A \leftarrow L_1, \ldots, L_n$ in $\text{ground}(P)$ either
    - $I \models L_1 \land \cdots \land L_n$ and $l(A) > l(L_i)$ for all $i$ or
    - exists $i$ s.t. $I \not\models L_i$ and $l(A) > l(L_i)$.

$d$ is $\Phi^{*}_{\omega}$-*accessible* if it is $\Phi^*$-accessible and $\gamma_{P} = \omega$.

* cf. Apt & Pedreschi 1993: acceptable programs
Generalized Fitting Construction

Neg\_P\*: predicates occurring negatively in P and all predicates on which they depend. P\^-: all ground clauses with head from Neg\_P\*.

K \in I\_P, then K\' is K restricted to predicates not in N.

Definitions: I \in I\_P and level map l fixed.

- \( f(K) = 0 \) if \( K \subseteq I \).
- \( f(K) = 2^{-\alpha} \) with \( \alpha \) least s.t. exists \( A \in K \setminus I \) with \( l(A) = \alpha \).
- \( u(K) = \max\{ f(K'), d(K', I) \} \).
- \( \varrho(J, K) = \max\{ d(J, K), u(J), u(K) \} \)

P is \( \Phi^*_\omega \)-accessible:

- \( (I\_P, \varrho) \) complete d-metric.
- \( T\_P \) contraction.
- \( T\_P \) has unique fixed point.
- \( P \) has unique supported model \( M \).
- \( T^n\_P(K) \to M \) in the Cantor topology on \( I\_P \) (for all \( K \in I\_P \)).

* cf. Fitting 1994
**Dislocated GUMs**

\((X, \varrho)\) *dislocated gum (d-gum)*:

- \(\varrho\) satisfies all conditions of a gum *except*
  - \(\varrho(x, x) = 0\) for all \(x \in X\).

Remaining notions as in the gum case.

\((X, \varrho)\) spherically complete d-gum,

\(f : X \to X\) strictly contracting.

Then \(f\) has a unique fixed point.

\(P\) \(\Phi^*\)-accessible:

- \((I_P, \varrho)\) spherically complete d-gum.
- \(T_P\) strictly contracting.
- \(T_P\) has unique fixed point.
- \(P\) has unique supported model \(M\).
- \(M\) can be obtained as the limit of a transfinite iterative process involving \(T_P\) and the Cantor topology on \(I_P\).
Another Application

$P$ is $\Phi$-accessible if $I$ model for $P$ and $l$ level map s.t. each $A \in B_P$ satisfies either (i) or (ii).

(i) Exists $A \leftarrow L_1, \ldots, L_n$ in ground($P$) s.t. $I \models L_1 \land \cdots \land L_n$ and $l(A) > l(L_i)$ for all $i$.

(ii) For each $A \leftarrow L_1, \ldots, L_n$ in ground($P$) exists $i$ with $I \not\models L_i$, $I \not\models A$, $l(A) > l(L_i)$.

$\star$ $P$ is $\Phi$-accessible iff $P$ has total model under Fitting-semantics (Fitting 1985).

Define $\delta(J, K) = \max\{d(J, I), d(K, I)\}$.

- $(I_P, \delta)$ spherically complete d-gum.
- $T_P$ strictly contracting.
- $M$ can be obtained as the limit of a transfinite iterative process involving $T_P$ and the Cantor topology on $I_P$. 
Discussion

• Understanding nonmonotonic reasoning.
  Rounds & Zhang 199x

• Exploring the “space” of all logic programs.

• Extensions to uncertain reasoning?
  van Emden 1986
  Mateis 1999

• Connections to topological dynamics.
  Seda & Hitzler 1997/8

• Relationships to artificial neural networks.
  Hölldobler et al. 199x
  Hitzler & Seda 2000
Appendix: Atomic Topology $Q$ on $I_P$

$B_P$ countable
then $Q$ homeomorphic to Cantor set.

Equivalent characterizations:

- Product topology on $2^{B_P}$
  where $2 = \{0, 1\}$ carries discrete topology.
- Subbase $\{G(L) \mid L \text{ literal}\}$,
  $G(L) = \{I \in I_P \mid I \models L\}$.
- $I_n \to I$ if
  each $A \in I$ is eventually in $I_n$ and
  each $A \notin I$ is eventually not in $I_n$.

For the transfinite iterative processes above:

For limit ordinal $\alpha$, set $I_\alpha$ to be set of all
$A \in B_P$ which are eventually in $(I_\beta)_{\beta < \alpha}$.

For successor ordinal $\alpha$, set $I_\alpha = T_P(I_{\alpha - 1})$.

Transfinite sequence obtained converges in $Q$.

* Batarekh & Subrahmanian 1989:
  Query Topology
Appendix: General Version of MPC&R

$(X, \varnothing)$ spherically complete d-gum.

$f : X \to X$ non-expanding

$(d(f(x), f(y)) \leq d(x, y)$ for all $x, y \in X$)

and $f$ strictly contracting on orbits

$(d(f^2(x), f(x)) < d(f(x), x)$

for all $x \in X$ with $x \neq f(x)$).

Then $f$ has fixed point.

$f$ strictly contracting then fixed point is unique.

- Also generalizes theorem by Khamsi, Kreinovich & Misane 1993.
- Constructive proof for applied special case possible.

Level mappings to rank functions:

$I \in I_P$ finite set (i.e. compact element of $I_P$).

$r(I) = \max\{l(A) \mid A \in I\}$. 
Appendix: Metrics and D-metrics

\[(X, d)\] complete ultrametric.
\[u : X \to \mathbb{R}_0^+\] continuous.

Then \[\varrho(x, y) = \max\{d(x, y), u(x), u(y)\}\]
is complete d-ultrametric.

\[(X, \varrho)\] complete d-metric.
\[d(x, y) = 0\text{ if } x = y,\]
\[d(x, y) = \varrho(x, y)\text{ otherwise.}\]

Then \(d\) complete metric.

If \(f\) contraction in \(\varrho\)
then \(f\) contraction in \(d\).

- Can prove the theorem of Matthews from Banach theorem.

- Anologous results hold for d-gums/gums.