Integrating Logic Programs and Connectionist Systems

A Constructive Approach

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Outline

Motivation

Logic Programs (LP)

- well-defined semantics
- human-readable
- human-writable

Connectionist Systems (CS)

- robust
- adaptive
- trainable

Goal:

- Integrate both paradigms in order to exploit all advantages

One step towards achieving this goal:

- Transform LP into CS

What we have so far:

- Constructions for Propositional LP
- Non-constructive proofs for First-Order LP

In this work:

- Constructions for First-Order LP

A Simple Example

- A Logic Program P

$even(0).$	% 0 is an even number
$even(s(X)) \leftarrow not\ even(X).$	% the successor of a
	% non-even X is even

- The Herbrand Base $\mathcal{B}_P$ and some Interpretations

$$\begin{aligned} \mathcal{B}_P &= \{even(0), even(s(0)), even(s^2(0)), \ldots\} \\ I_1 &= \{even(0), even(s(0))\} \\ I_2 &= \{even(0), even(s^3(0)), even(s^4(0)), even(s^5(0)), \ldots\} \end{aligned}$$

- The Single-Step Operator or Meaning Function T_P

$$\begin{aligned} I_1 \overset{T_P}{\longmapsto} I_2 \quad &\overset{T_P}{\longmapsto} \quad \{even(0), even(s^2(0)), even(s^3(0))\} \\ \overset{T_P}{\longmapsto} \ldots \quad &\overset{T_P}{\longmapsto} \quad \{even(0), even(s^2(0)), even(s^4(0)), \\ &\qquad even(s^6(0)), even(s^8(0)), even(s^{10}(0)), \ldots\} \end{aligned}$$

Embedding T_P in $\mathbb{R}$

- Enumerate $\mathcal{B}_P$ using $\|\cdot\| : \mathcal{B}_P \to \mathbb{N} \setminus \{0\}$

$$\|even(s^n(0))\| := n+1$$

- Embed $I \in \mathcal{I}_P$ into $\mathbb{R}$ using $R(I) := \sum_{A \in I} 3^{-\|A\|}$

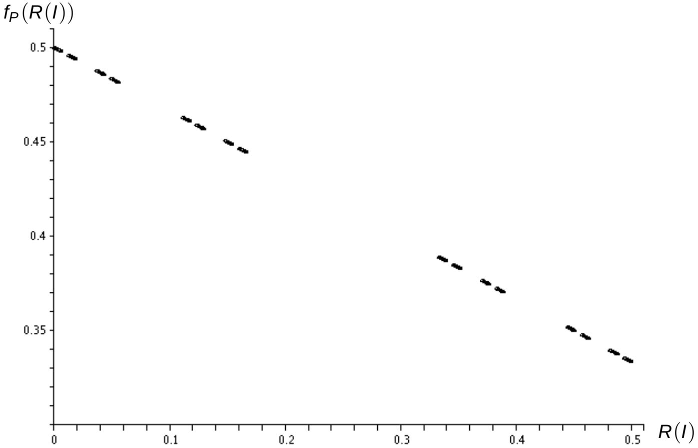
$$R\left(\{even(0), even(s^2(0))\}\right) = 0.\underset{even(0)}{1}\ \underset{even(s(0))}{0}\ \underset{even(s^2(0))}{1}\ \underset{even(s^3(0))}{0}\ \underset{even(s^4(0))}{0}\ \ldots_3$$

- Embed T_P into $\mathbb{R}$:

$$\begin{array}{ccc} I \in \mathcal{I}_P & \xrightarrow[T_P]{} & I' \in \mathcal{I}_P \\ \uparrow R^{-1} & & \downarrow R \\ x \in D_f & \overset{f_P}{\dashrightarrow} & x' \in D_f \end{array}$$

where $D_f := \{R(I) | I \in \mathcal{I}_P\}$

Embedding of the Example Program



In general, the graph is more complicated and not on a straight line!

Idea for Approximating f_P

- Goal: approximate f_P (the embedded T_P) up to ε
- Consider $x, x' \in D_f$:

$$x = 0.\underbrace{001010110 10}_{l \text{ digits are equal}}000000\ldots_3$$

$$x' = 0.\overbrace{00101011010}111111\ldots_3$$

Maximum difference $\delta_l := \sum_{i>l} 3^{-i} = \frac{1}{3^l \cdot 2}$

- Greatest relevant output level $o_\varepsilon := \min \left\{ n \in \mathbb{N} \middle| \delta_n < \varepsilon \right\}$
- Assume $T_{P'}$ and T_P agree on all atoms of level $\leq o_\varepsilon$

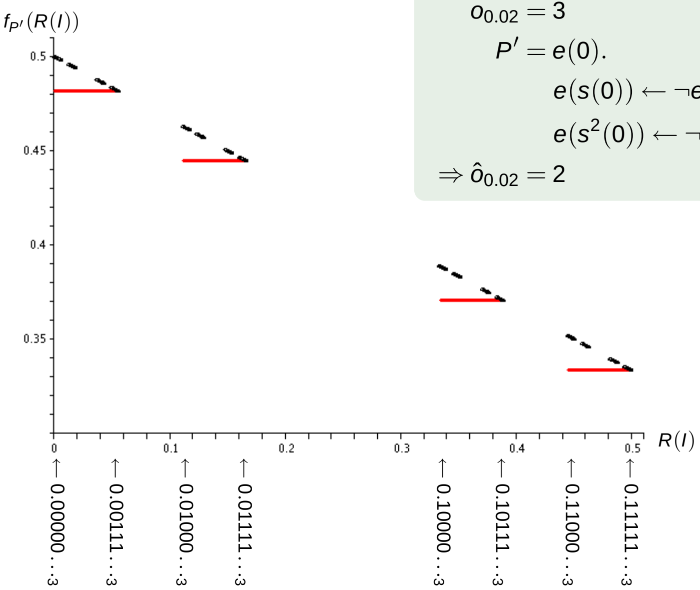
  $\Rightarrow f_{P'}$ and f_P agree on the first o_ε digits

  $\Rightarrow f_{P'}$ approximates f_P up to ε

The Instance of P up to o_ε

- Goal: find P' such that $T_{P'}$ and T_P agree on atoms of level $\leq o_\varepsilon$
- Inclusion of A in $T_P(I)$ depends only on clauses with head A
- $P' := \left\{A \leftarrow B \in \mathcal{G}(P) \middle| \|A\| \leq o_\varepsilon\right\}$
  where $\mathcal{G}(P) :=$ set of all ground instances of clauses from P
- P' is finite if P is covered, i.e. if there are no local variables
- Greatest relevant input level
  $\hat{o}_\varepsilon := \max\left\{\|L\| \middle| L \text{ is body literal of some clause in } P'\right\}$
- $T_{P'}$ depends only on atoms of level $\leq \hat{o}_\varepsilon$
  $\Rightarrow f_{P'}$ depends only on the first $\hat{o}_\varepsilon$ digits
  $\Rightarrow f_{P'}$ is constant for all inputs which agree on first $\hat{o}_\varepsilon$ digits
  $\Rightarrow f_{P'}$ consists of finitely many constant pieces

Our Example with $\varepsilon = 0.02$

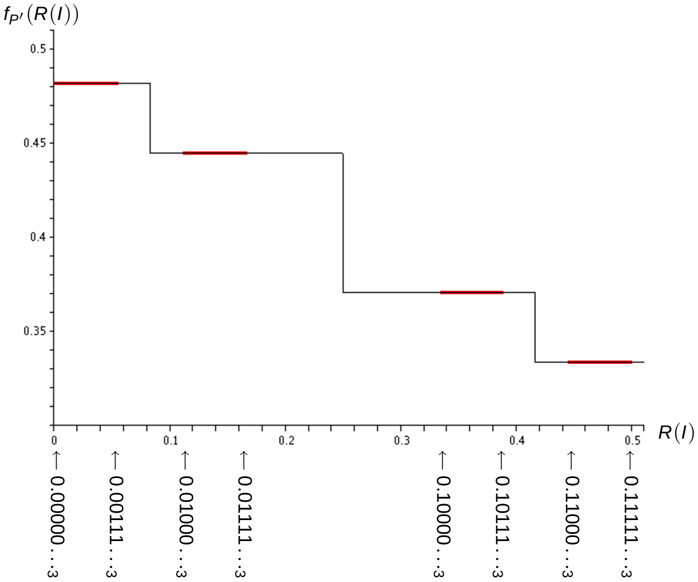


$$o_{0.02} = 3$$

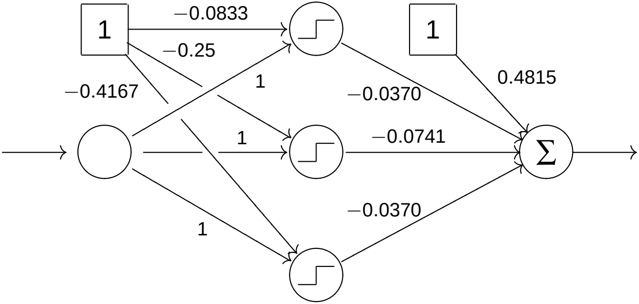
$$P' = e(0).$$
$$e(s(0)) \leftarrow \neg e(0).$$
$$e(s^2(0)) \leftarrow \neg e(s(0)).$$

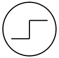
$$\Rightarrow \hat{o}_{0.02} = 2$$

Building a CS with Step Activation Functions

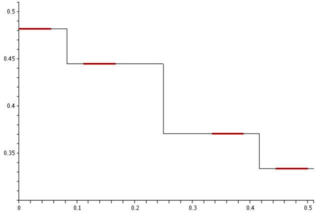


The Resulting CS



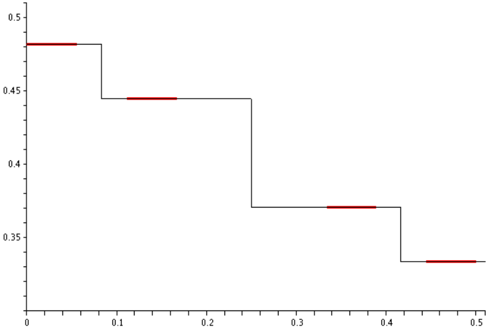
Each  computes

- 0, if weighted sum of inputs ≤ 0
- 1, otherwise



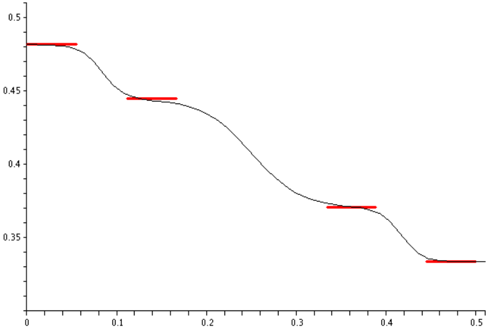
Building a CS with Sigmoidal Activation Functions

Approximate the step functions



Building a CS with Sigmoidal Activation Functions

Approximate the step functions by sigmoidals



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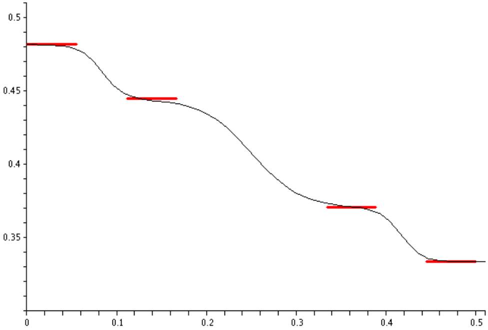

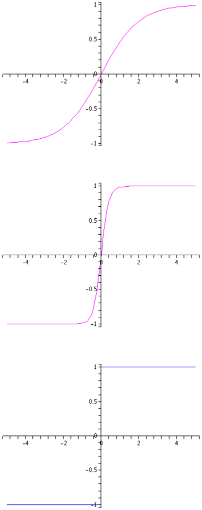

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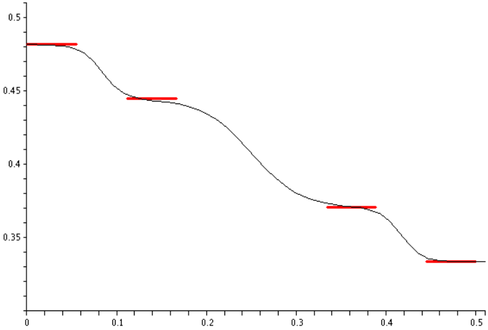
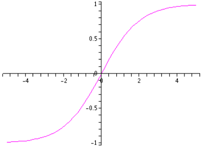
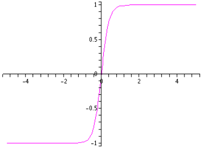
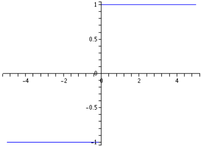
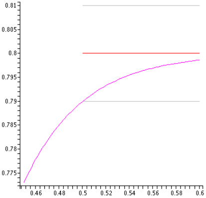








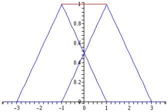


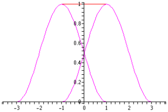
Divide ε into ε′ for P' and ε″ for the sigmoidals

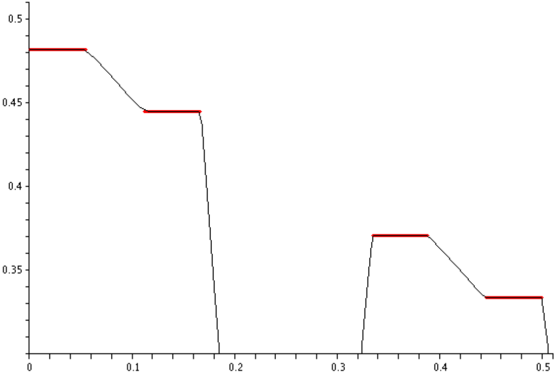
The closest constant piece yields the zoom-out factor

A CS with Triangle or Raised-Cosine Activation Functions

Describe each constant piece by two triangles or raised cosines:

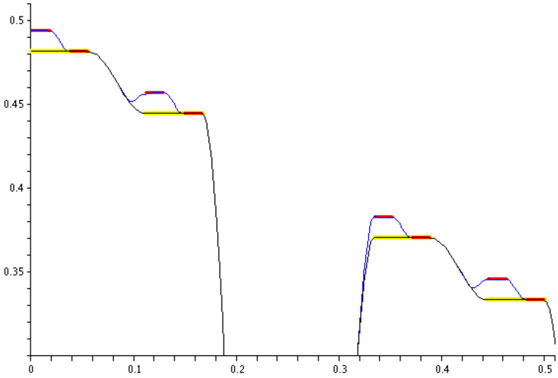






Refining an Existing Network

- Decreasing ε will only add clauses to P'
- Consequence:
  - Constant pieces may be divided into smaller pieces
  - Some parts may be raised
- For $\varepsilon = 0.007$, we get:



Conclusions and Problems

What we had before:

- Methods to construct CS for propositional LP
- Non-constructive proofs for the existence of CS approximating first-order LP

New results:

- Methods for constructing CS approximating first-order LP
- Method for iterative refinement

Problem:

- Floating point precision in real computers is very limited, so we can represent only few atoms

Possible remedy:

- Distribute representation on several input/output nodes

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Thank you for your attention.