Proving the Validity of an Argument

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Introduction

Proving Validity: Examples

The Notion of Proof Rules

Proving Validity: More Examples

Fitch

Reasoning about Identity
In Fitch format, an argument takes the form

\[
\begin{array}{c}
P_1 \\
\cdots \\
P_n \\
Q
\end{array}
\]

- premises
- conclusion

Such an argument is valid if conclusion \( Q \) is true whenever premises \( P_1 \ldots P_n \) are.

A valid argument if sound if the premises are true.
To show that an argument is not valid, use counterexamples: a world where the premises are true, but conclusion false.

How to show that an argument is valid? (topic of Section 2.2-2.4)

- naive approach: consider all worlds where premise is true, and show that also conclusion is true.
- feasible approach: construct a proof
Informal reasoning:

We are told that $b$ is to the right of $c$. So $c$ must be to the left of $b$, since right of and left of are inverses of one another. And since $b = d$, $c$ is left of $d$, by the indiscernibility of identicals. But we are also told that $d$ is left of $e$, and consequently $c$ is to the left of $e$, by the transitivity of left of. This is our desired conclusion.
A Valid Argument

RightOf(b,c)  
LeftOf(d,e)  
b = d  
LeftOf(c,e)  why?

Informal reasoning: (p.52)
We are told that b is to the right of c. So c must be to the left of b, since right of and left of are inverses of one another. And since b = d, c is left of d, by the indiscernibility of identicals. But we are also told that d is left of e, and consequently c is to the left of e, by the transitivity of left of. This is our desired conclusion.
We establish a series of intermediate results:

1. RightOf(b,c)
2. LeftOf(d,e)
3. b = d
4. LeftOf(c,b) from 1, since LeftOf and RightOf are inverses
5. LeftOf(c,d) from 4 and 3, using identity elimination
6. LeftOf(c,e) from 5 and 2, since LeftOf is transitive
Identity Elimination (\(\equiv\) Elim)

- If \(b\) is a cube and \(b\) equals \(c\), then also \(c\) is a cube
- If John is happy and John is the father of Max, then the father of Max is happy

In general (cf. p. 56)

\[
\begin{array}{c}
P(t) \\
\vdots \\
t = u \\
\vdots \\
\hline \\
P(u)
\end{array}
\]
A Valid Arithmetic Argument

1. $x > 2$

2. $2 > 0$

3. $x > 0$ from 1 and 2, since $>$ is transitive

4. $x \cdot x > 2 \cdot x$ from 1 and 3, using law of arithmetic: multiplying by positive number preserves inequalities

5. $2 \cdot x = x + x$ fact

$x \cdot x > x + x$ why?
A Valid Arithmetic Argument

1. \( x > 2 \)
2. \( 2 > 0 \)  fact
3. \( x > 0 \)  from 1 and 2, since \( > \) is transitive
4. \( x \cdot x > 2 \cdot x \)  from 1 and 3, using law of arithmetic: multiplying by positive number preserves inequalities
5. \( 2 \cdot x = x + x \)  fact
6. \( x \cdot x > x + x \)  from 4 and 5, using \( = \text{Elim} \)
Proof in Fitch

Converting previous proof into Fitch (software package):

1. RightOf(b,c)
2. LeftOf(d,e)
3. b = d
4. LeftOf(c,b) from 1, since LeftOf and RightOf are inverses
5. LeftOf(c,d) from 4 and 3, using identity elimination
6. LeftOf(c,e) from 5 and 2, since LeftOf is transitive
Proof in Fitch

Converting previous proof into Fitch (software package):

1. RightOf(b,c)
2. LeftOf(d,e)
3. b = d
4. LeftOf(c,b)  \textbf{Ana Con:} 1
5. LeftOf(c,d) from 4 and 3, using identity elimination
6. LeftOf(c,e) from 5 and 2, since LeftOf is transitive
Proof in Fitch

Converting previous proof into Fitch (software package):

1. \text{RightOf}(b,c)
2. \text{LeftOf}(d,e)
3. \text{b} = \text{d}
4. \text{LeftOf}(c,b) \quad \textbf{Ana Con: 1}
5. \text{LeftOf}(c,d) \quad \textbf{= Elim: 4, 3}
6. \text{LeftOf}(c,e) \quad \text{from 5 and 2, since \text{LeftOf} is transitive}
Proof in Fitch

Converting previous proof into Fitch (software package):

1. RightOf(b,c)
2. LeftOf(d,e)
3. b = d
4. LeftOf(c,b)   \textbf{Ana Con}: 1
5. LeftOf(c,d)   \textbf{Elim}: 4, 3
6. LeftOf(c,e)   \textbf{Ana Con}: 5, 2
Limitations on Fitch

- Fitch is tuned for Tarski’s World, with Ana Con modeling the laws of the block world.
- Fitch is not tuned for arithmetic. Thus Fitch cannot handle the proof that $x \cdot x > x + x$. 

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- System F is not tuned for Tarski’s world, and therefore has no special proof rules for the predicates there, like LeftOf.
- But since the identity relation “=” is used in all domains of discourse, system F has special rules for that predicate.
# System F, vs. Fitch

<table>
<thead>
<tr>
<th>what is?</th>
<th>system F (p.54)</th>
<th>Fitch (p.58)</th>
</tr>
</thead>
<tbody>
<tr>
<td>application domain</td>
<td>formal system</td>
<td>software package</td>
</tr>
<tr>
<td></td>
<td>general purpose</td>
<td>tuned for Tarski’s world</td>
</tr>
<tr>
<td>Proof rules:</td>
<td>$\equiv \text{Elim}, \equiv \text{Intro}$</td>
<td>$\equiv \text{Elim}, \equiv \text{Intro}$</td>
</tr>
<tr>
<td></td>
<td>$\land \text{Elim}, \land \text{Intro}$</td>
<td>$\land \text{Elim}, \land \text{Intro}$</td>
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<tr>
<td></td>
<td>etc.</td>
<td>etc.</td>
</tr>
<tr>
<td>shortcuts</td>
<td>can be added</td>
<td></td>
</tr>
<tr>
<td>specific</td>
<td>can be added</td>
<td></td>
</tr>
</tbody>
</table>

- Taut Con, FO Con
- Ana Con
(encodes block world laws)
Identity Introduction (\(=\) Intro)

Rule described p.55 is amazingly simple:

\[\vartriangleleft | t = t\]

This says that the identity relation is reflexive.
Properties of Identity

**Symmetry**

\[
\begin{align*}
  a &= b \\
  b &= a
\end{align*}
\]
Properties of Identity

Symmetry

\[
\begin{array}{c}
a = b \\
\hline
b = a
\end{array}
\]

Transitivity

\[
\begin{array}{c}
a = b \\
\hline
b = c
\end{array}
\]

\[
\begin{array}{c}
a = c
\end{array}
\]
Properties of Identity

Symmetry

1. \(a = b\)
2. \(a = a\) \(=\) Intro
3. \(b = a\) \(=\) Elim: 2, 1

Transitivity

\[
\begin{align*}
\frac{a = b}{b = c} \\
\frac{b = c}{a = c}
\end{align*}
\]
Properties of Identity

Symmetry

1. $a = b$
2. $a = a = \text{Intro}$
3. $b = a = \text{Elim: } 2, 1$

Transitivity

1. $a = b$
2. $b = c$
3. $a = c = \text{Elim: } 1, 2$