Boolean Connectives

Torben Amtoft
Kansas State University

Agenda

- Chapter 1 introduced basic FOL
  (one main aim of book)
- Chapter 2 introduced notion of logical consequence
  (other main aim of book)
- Chapter 3 introduces more features of FOL

Boolean Connectives

Recall that an atomic sentence is a predicate applied to one or more terms:

```
Older(father(max),max)
```

We now extend FOL with the boolean connectives:

- and, to be written $\wedge$
- or, to be written $\vee$
- not, to be written $\neg$.

Negation ("not")

Truth table:

P	$\neg P$
true	false
false	true

- Symbol $\neg$ is not standard (cf. p. 91); in emails and on the web I'll write ~.
- $\neg\neg P$ is equivalent to P unlike English, where double negation emphasizes: *it doesn't make no difference*; *there will be no nothing*
- $\neg\texttt{LeftOf}(\texttt{a}, \texttt{b})$ is not equivalent to $\texttt{RightOf}(\texttt{a}, \texttt{b})$

Conjunction ("and")

P	Q	$P \wedge Q$
true	true	true
true	false	false
false	true	false
false	false	false

- in emails and on the web I may write /\ or ^
- English sentences translated using $\wedge$ may
  - not use "and"

    Max is a tall man `Tall(max)` $\wedge$ `Man(max)`
  - carry temporal implications

    Max went home and went to sleep
  - be expressed using other connectives

    Max was home but Claire was not

Disjunction ("or")

P	Q	$P \vee Q$
true	true	true
true	false	true
false	true	true
false	false	false

- in emails and on the web I may write \/ or `v`.
- the interpretation is "inclusive", not "exclusive": true $\vee$ true $=$ true.
- In English, the default is often "exclusive", as when a waiter offers *soup or salad*
- We can express exclusive or (p. 75):

Disjunction ("or")

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- We can express exclusive or (p. 75): $(P \vee Q) \wedge \neg(P \wedge Q)$
- We can also encode "neither nor": $\neg(P \vee Q)$

Sentences

A sentence P is thus given by

- if P is an atomic sentence then P is also a sentence;
- if P_1 and P_2 are sentences then $P_1 \wedge P_2$ is a sentence;
- if P_1 and P_2 are sentences then $P_1 \vee P_2$ is a sentence;
- if P is a sentence then $\neg P$ is a sentence.

This can be written in "Backus-Naur" notation:

$$
\begin{array}{lcl}
P & ::= & \text{atomic sentence} \\
 & \mid & P \wedge P \\
 & \mid & P \vee P \\
 & \mid & \neg P
\end{array}
$$

Resolving Ambiquity

	expression	how to read it	how not to read it
Algebra	$3 + 4 \times 5$	$3 + (4 \times 5) = 23$	$(3 + 4) \times 5 = 35$
	$3 \times 4 + 5$	$(3 \times 4) + 5$	$3 \times (4 + 5)$

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Boolean Algebra	interpretation I	interpretation II
true $\vee$ false $\wedge$ false	true $\vee$ (false $\wedge$ false) evaluates to true	(true $\vee$ false) $\wedge$ false evaluates to false

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> *Parentheses must be used whenever ambiguity would result from their omission*

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Negation binds tightly: $\neg P \wedge Q$ is not equivalent to $\neg(P \wedge Q)$.

Ambiguity in English

Consider the phrase

you can have soup or salad and pasta.

If the intended meaning is "soup or (salad and pasta)":

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or

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you can have soup or salad, and pasta

or

you can have pasta and either soup or salad

The Game in Tarski's World

- Given sentence $P = \texttt{Cube(c)} \vee \texttt{Cube(d)}$.
- Given world where `c` is a cube but `d` is not.

We	Opponent
P is false in this world	
	So `c` is not a cube?
Eh... I admit defeat	

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OK, P is true in this world	
	Because c is a cube or because d is?
Because d is a cube	
	You lost but could have won

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Because d is a cube	
	You lost but could have won
OK, because c is a cube	
	You won (finally!)

More about the Game

- Given sentence $P = \texttt{Cube(a)} \lor \neg\texttt{Cube(a)}$.

We	Opponent
P is true in this world	
	Because a is a cube or because a is not a cube?
Eh. . . I don't know but P will always be true!	
	Please answer my question!

- Who won the game???