Boolean Connectives

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Agenda

- **Chapter 1** introduced basic FOL
  (one main aim of book)
- **Chapter 2** introduced notion of logical consequence
  (other main aim of book)
- **Chapter 3** introduces more features of FOL
Recall that an atomic sentence is a predicate applied to one or more terms:

\[ \text{Older}(\text{father}(\text{max}), \text{max}) \]

We now extend FOL with the boolean connectives:

- **and**, to be written $\land$
- **or**, to be written $\lor$
- **not**, to be written $\neg$. 
Negation ("not")

Truth table:

<table>
<thead>
<tr>
<th>$P$</th>
<th>$\neg P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>true</td>
<td>false</td>
</tr>
<tr>
<td>false</td>
<td>true</td>
</tr>
</tbody>
</table>

- Symbol $\neg$ is not standard (cf. p. 91); in emails and on the web I'll write $\sim$.

- $\neg \neg P$ is equivalent to $P$
  unlike English, where double negation emphasizes: *it doesn’t make no difference; there will be no nothing*

- $\neg \text{LeftOf}(a, b)$ is not equivalent to $\text{RightOf}(a, b)$
Conjunction ("and")

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \land Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>true</td>
<td>true</td>
<td>true</td>
</tr>
<tr>
<td>true</td>
<td>false</td>
<td>false</td>
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- in emails and on the web I may write $\land$ or $\&$
- English sentences translated using $\land$ may not use "and"
  
  Max is a tall man $\land$ Tall(max) $\land$ Man(max)
- carry temporal implications
  
  Max went home and went to sleep
- be expressed using other connectives
  
  Max was home but Claire was not
Disjunction ("or")

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \lor Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>true</td>
<td>true</td>
<td>true</td>
</tr>
<tr>
<td>true</td>
<td>false</td>
<td>true</td>
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- in emails and on the web I may write \(\setminus/\) or \(\lor\).
- the interpretation is "inclusive", not "exclusive":
  \[true \lor true = true\]
- In English, the default is often "exclusive", as when a waiter offers *soup or salad*
- We can express exclusive or (p. 75):
Disjunction ("or")

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- In emails and on the web I may write \( \lor \) or \( \lor \).
- The interpretation is "inclusive", not "exclusive": 
  \[ \text{true} \lor \text{true} = \text{true}. \]
- In English, the default is often "exclusive", as when a waiter offers *soup or salad*
- We can express exclusive or (p. 75): \((P \lor Q) \land \neg(P \land Q)\)
- We can also encode "neither nor": \(\neg(P \lor Q)\)
A sentence $P$ is thus given by

- if $P$ is an atomic sentence then $P$ is also a sentence;
- if $P_1$ and $P_2$ are sentences then $P_1 \land P_2$ is a sentence;
- if $P_1$ and $P_2$ are sentences then $P_1 \lor P_2$ is a sentence;
- if $P$ is a sentence then $\neg P$ is a sentence.

This can be written in “Backus-Naur” notation:

$$P ::= \text{atomic sentence}$$

$$| \quad P \land P$$

$$| \quad P \lor P$$

$$| \quad \neg P$$
## Resolving Ambiguity

<table>
<thead>
<tr>
<th>Expression</th>
<th>How to Read It</th>
<th>How Not to Read It</th>
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<tbody>
<tr>
<td>$3 + 4 \times 5$</td>
<td>$3 + (4 \times 5) = 23$</td>
<td>$(3 + 4) \times 5 = 35$</td>
</tr>
<tr>
<td>$3 \times 4 + 5$</td>
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| **Boolean Algebra**               | interpretation I | interpretation II       |                            |
| true $\lor$ false $\land$ false   | true $\lor$ (false $\land$ false) evaluates to true | (true $\lor$ false) $\land$ false evaluates to false |
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- In the literature, I is often chosen (as $\land$ is considered “multiplication” and $\lor$ is considered “addition”).
- In the textbook, neither I or II is chosen, instead (p. 80):

  > Parentheses must be used whenever ambiguity would result from their omission.
Resolving Ambiguity

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- In the textbook, neither I or II is chosen, instead (p. 80):

  *Parentheses must be used whenever ambiguity would result from their omission*

Negation binds tightly: $\neg P \land Q$ is not equivalent to $\neg(P \land Q)$. 
Ambiguity in English

Consider the phrase

\[ you \ can \ have \ soup \ or \ salad \ and \ pasta. \]

If the intended meaning is “soup or (salad and pasta)”: 

Consider the phrase

\textit{you can have soup or salad and pasta.}

If the intended meaning is “soup or (salad and pasta)”:

\textit{you can have soup or both salad and pasta}

If the intended meaning is “(soup or salad) and pasta”:
Ambiguity in English

Consider the phrase

\textit{you can have soup or salad and pasta.}

If the intended meaning is “soup or (salad and pasta)”:

\textit{you can have soup or both salad and pasta}

If the intended meaning is “(soup or salad) and pasta”:

\textit{you can have soup or salad, and pasta}

or
Ambiguity in English

Consider the phrase

\[ \text{you can have soup or salad and pasta.} \]

If the intended meaning is “soup or (salad and pasta)”:

\[ \text{you can have soup or both salad and pasta} \]

If the intended meaning is “(soup or salad) and pasta”:

\[ \text{you can have soup or salad, and pasta} \]

or

\[ \text{you can have pasta and either soup or salad} \]
The Game in Tarski’s World

- Given sentence $P = \text{Cube}(c) \lor \text{Cube}(d)$.
- Given world where $c$ is a cube but $d$ is not.

We: $P$ is false in this world

Opponent: So $c$ is not a cube?

Eh... I admit defeat
Given sentence $P = \text{Cube}(c) \lor \text{Cube}(d)$.  
Given world where $c$ is a cube but $d$ is not.  

We

Opponent

$P$ is false in this world

So $c$ is not a cube?

Eh... I admit defeat

OK, $P$ is true in this world

Because $c$ is a cube or because $d$ is?

Because $d$ is a cube

You lost but could have won
The Game in Tarski’s World

- Given sentence $P = \text{Cube}(c) \lor \text{Cube}(d)$.
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<tr>
<td>$P$ is false in this world</td>
<td>So $c$ is not a cube?</td>
</tr>
<tr>
<td>Eh. . . I admit defeat</td>
<td></td>
</tr>
<tr>
<td>OK, $P$ is true in this world</td>
<td>Because $c$ is a cube or because $d$ is?</td>
</tr>
<tr>
<td>Because $d$ is a cube</td>
<td>You lost but could have won</td>
</tr>
<tr>
<td>OK, because $c$ is a cube</td>
<td>You won (finally!)</td>
</tr>
</tbody>
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More about the Game

- Given sentence $P = \text{Cube}(a) \lor \neg\text{Cube}(a)$.

<table>
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<th>Opponent</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$ is true in this world</td>
<td>Because $a$ is a cube or because $a$ is not a cube?</td>
</tr>
<tr>
<td>Eh... I don't know but $P$ will always be true!</td>
<td>Please answer my question!</td>
</tr>
</tbody>
</table>

- Who won the game???