Proving the Validity of an Argument

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Introduction

Proving Validity: Examples

The Notion of Proof Rules

Proving Validity: More Examples

Fitch

Reasoning about Identity
In Fitch format, an argument takes the form

\[
\begin{array}{c|c}
P_1 & \text{premises} \\
\vdots & \\
P_n & \\
\hline
Q & \text{conclusion}
\end{array}
\]

▶ Such an argument is valid if conclusion \( Q \) is true whenever premises \( P_1 \ldots P_n \) are.
▶ A valid argument if sound if the premises are true.
Proving and Disproving Validity

- To show that an argument is not valid, use counterexamples: a world where the premises are true, but conclusion false.

- How to show that an argument is valid? (topic of Section 2.2-2.4)
  - naive approach: consider all worlds where premise is true, and show that also conclusion is true.
  - feasible approach: construct a proof
A Valid Argument

\[
\begin{align*}
\text{RightOf}(b,c) \\
\text{LeftOf}(d,e) \\
b = d \\
\text{LeftOf}(c,e) & \quad \text{why?}
\end{align*}
\]
A Valid Argument

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\end{align*}
\]

**Informal reasoning:** (p.52)
We are told that \( b \) is to the right of \( c \). So \( c \) must be to the left of \( b \), since \textit{right of} and \textit{left of} are inverses of one another. And since \( b = d \), \( c \) is left of \( d \), by the indiscernibility of identicals. But we are also told that \( d \) is left of \( e \), and consequently \( c \) is to the left of \( e \), by the transitivity of \textit{left of}. This is our desired conclusion.
A Formal Proof

We establish a series of intermediate results:

1. RightOf(b,c)
2. LeftOf(d,e)
3. b = d
4. LeftOf(c,b)  from 1, since LeftOf and RightOf are inverses
5. LeftOf(c,d)  from 4 and 3, using identity elimination
6. LeftOf(c,e)  from 5 and 2, since LeftOf is transitive
Identity Elimination (= Elim)

- If $b$ is a cube and $b$ equals $c$, then also $c$ is a cube
- If John is happy and John is the father of Max, then the father of Max is happy

In general (cf. p. 56)

\[
\begin{array}{c}
P(t) \\
\ldots \\
t = u \\
\ldots \\
\triangleright \\
P(u)
\end{array}
\]
A Valid Arithmetic Argument

\[
x > 2
\]

\[
x \cdot x > x + x \quad \text{why?}
\]
A Valid Arithmetic Argument

1. $x > 2$
2. $2 > 0$  
   fact
3. $x > 0$  
   from 1 and 2, since $>$ is transitive
4. $x \cdot x > 2 \cdot x$  
   from 1 and 3, using law of arithmetic: 
   multiplying by positive number preserves inequalities
5. $2 \cdot x = x + x$  
   fact
6. $x \cdot x > x + x$  
   from 4 and 5, using $=$ Elim
System F, vs. Fitch

<table>
<thead>
<tr>
<th></th>
<th>System F (p.54)</th>
<th>Fitch (p.58)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>what is?</strong></td>
<td>formal system</td>
<td>software package</td>
</tr>
<tr>
<td><strong>application domain</strong></td>
<td>general purpose</td>
<td>tuned for Tarski’s world</td>
</tr>
<tr>
<td><strong>Proof rules:</strong></td>
<td>개최,  소개, 설명</td>
<td>개최,  소개, 설명</td>
</tr>
<tr>
<td></td>
<td>인트로,  인트로, 설명</td>
<td>Elim,  Intro,  Elim,  Intro</td>
</tr>
<tr>
<td></td>
<td>곱셈,  인트로,  설명</td>
<td>Elim,  Intro</td>
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<td></td>
<td>등등.</td>
<td>등등.</td>
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<tr>
<td><strong>shortcuts</strong></td>
<td>can be added</td>
<td>Taut Con, FO Con</td>
</tr>
<tr>
<td><strong>specific</strong></td>
<td>can be added</td>
<td>Ana Con (encodes block world laws)</td>
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</tbody>
</table>

Proving the Validity of an Argument
Proof in Fitch

Converting previous proof into Fitch:

1. RightOf(b,c)
2. LeftOf(d,e)
3. b = d
4. LeftOf(c,b)  from 1, since LeftOf and RightOf are inverses
5. LeftOf(c,d)  from 4 and 3, using identity elimination
6. LeftOf(c,e)  from 5 and 2, since LeftOf is transitive
Proof in Fitch

Converting previous proof into Fitch:

1. $\text{RightOf}(b,c)$
2. $\text{LeftOf}(d,e)$
3. $b = d$
4. $\text{LeftOf}(c,b)$ \hspace{1cm} \textbf{Ana Con:} 1
5. $\text{LeftOf}(c,d)$ \hspace{1cm} from 4 and 3, using identity elimination
6. $\text{LeftOf}(c,e)$ \hspace{1cm} from 5 and 2, since $\text{LeftOf}$ is transitive
Proof in Fitch

Converting previous proof into Fitch:

1. RightOf(b,c)
2. LeftOf(d,e)
3. b = d
4. LeftOf(c,b)  \textbf{Ana Con}: 1
5. LeftOf(c,d)  \textbf{Elim}: 4, 3
6. LeftOf(c,e) from 5 and 2, since LeftOf is transitive
Proof in Fitch

Converting previous proof into Fitch:

1. RightOf(b,c)
2. LeftOf(d,e)
3. b = d
4. LeftOf(c,b) | Ana Con: 1
5. LeftOf(c,d) | = Elim: 4, 3
6. LeftOf(c,e) | Ana Con: 5, 2
Limitations on Fitch

- Fitch is tuned for Tarski’s World, with Ana Con modeling the laws of the block world.
- Fitch is not tuned for arithmetic. Thus Fitch cannot handle the proof that $x \cdot x > x + x$. 

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- System F is not tuned for Tarski’s world, and therefore has no special proof rules for the predicates there, like LeftOf.
- But since the identity relation “$=$” is used in all domains of discourse, system F has special rules for that predicate.
Identity Introduction (\(=\) Intro)

Rule described p.55 is amazingly simple:

\[
\vdash \quad t = t
\]

This says that the identity relation is reflexive.
Properties of Identity

Symmetry

\[
\begin{align*}
\text{a} & \equiv \text{b} \\
\text{b} & \equiv \text{a}
\end{align*}
\]
Properties of Identity

Symmetry

\[
\begin{align*}
\text{a} &= \text{b} \\
\text{b} &= \text{a}
\end{align*}
\]

Transitivity

\[
\begin{align*}
\text{a} &= \text{b} \\
\text{b} &= \text{c} \\
\text{a} &= \text{c}
\end{align*}
\]
Properties of Identity

**Symmetry**

1. $a = b$
2. $a = a$ = Intro
3. $b = a$ = Elim: 2, 1

**Transitivity**

<table>
<thead>
<tr>
<th>a = b</th>
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<tbody>
<tr>
<td>b = c</td>
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</tr>
</tbody>
</table>
Properties of Identity

Symmetry

1. a = b
2. a = a \Rightarrow \text{Intro}
3. b = a \Rightarrow \text{Elim: 2, 1}

Transitivity

1. a = b
2. b = c
3. a = c \Rightarrow \text{Elim: 1, 2}