Proving the Validity of an Argument

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Reminder

In Fitch format, an argument takes the form

P_1	
$\cdots$	premises
P_n	
Q	conclusion

- Such an argument is valid if conclusion Q is true whenever premises $P_1 \ldots P_n$ are.
- A valid argument if sound if the premises are true.

Proving and Disproving Validity

- To show that an argument is not valid, use counterexamples: a world where the premises are true, but conclusion false.
- How to show that an argument is valid?
  (topic of Section 2.2-2.4)
  - naive approach: consider all worlds where premise is true, and show that also conclusion is true.
  - feasible approach: construct a proof

A Valid Argument

```
RightOf(b,c)
LeftOf(d,e)
b = d
------------
LeftOf(c,e)     why?
```

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Informal reasoning: (p.52)
We are told that *b* is to the right of *c*. So *c* must be to the left of *b*, since *right of* and *left of* are inverses of one another. And since $b = d$, *c* is left of *d*, by the indiscernibility of identicals. But we are also told that *d* is left of *e*, and consequently *c* is to the left of *e*, by the transitivity of *left of*. This is our desired conclusion.

A Formal Proof

We establish a series of intermediate results:

1.	RightOf(b,c)	
2.	LeftOf(d,e)	
3.	b = d	
4.	LeftOf(c,b)	from 1, since LeftOf and RightOf are inverses
5.	LeftOf(c,d)	from 4 and 3, using identity elimination
6.	LeftOf(c,e)	from 5 and 2, since LeftOf is transitive

Identity Elimination (= Elim)

- If b is a cube and b equals c, then also c is a cube
- If John is happy and John is the father of Max, then the father of Max is happy

In general (cf. p. 56)

$$
\begin{array}{l|l}
 & P(t) \\
 & \ldots \\
 & t = u \\
 & \ldots \\
\triangleright & P(u)
\end{array}
$$

A Valid Arithmetic Argument

$x > 2$

$x \cdot x > x + x$ why?

A Valid Arithmetic Argument

1.	$x > 2$	
2.	$2 > 0$	fact
3.	$x > 0$	from 1 and 2, since $>$ is transitive
4.	$x \cdot x > 2 \cdot x$	from 1 and 3, using law of arithmetic: multiplying by positive number preserves inequalities
5.	$2 \cdot x = x + x$	fact
6.	$x \cdot x > x + x$	from 4 and 5, using **= Elim**

System F, vs. Fitch

	system F (p.54)	Fitch (p.58)
what is?	formal system	software package
application domain	general purpose	tuned for Tarski's world
Proof rules:		
general	**= Elim**, **= Intro**	**= Elim**, **= Intro**
	$\wedge$ **Elim**, $\wedge$ **Intro**	$\wedge$ **Elim**, $\wedge$ **Intro**
	etc.	etc.
shortcuts	can be added	**Taut Con**, **FO Con**
specific	can be added	**Ana Con**
		(encodes block world laws)

Proof in Fitch

Converting previous proof into Fitch:

1.	RightOf(b,c)	
2.	LeftOf(d,e)	
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Proof in Fitch

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1.	RightOf(b,c)	
2.	LeftOf(d,e)	
3.	b = d	
4.	LeftOf(c,b)	**Ana Con**: 1
5.	LeftOf(c,d)	from 4 and 3, using identity elimination
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Proof in Fitch

Converting previous proof into Fitch:

1.	`RightOf(b,c)`	
2.	`LeftOf(d,e)`	
3.	`b = d`	
4.	`LeftOf(c,b)`	**Ana Con**: 1
5.	`LeftOf(c,d)`	**= Elim**: 4, 3
6.	`LeftOf(c,e)`	from 5 and 2, since `LeftOf` is transitive

Proof in Fitch

Converting previous proof into Fitch:

1. RightOf(b,c)	
2. LeftOf(d,e)	
3. b = d	
4. LeftOf(c,b)	**Ana Con**: 1
5. LeftOf(c,d)	**= Elim**: 4, 3
6. LeftOf(c,e)	**Ana Con**: 5, 2

Limitations on Fitch

- Fitch is tuned for Tarski's World,
  with **Ana Con** modeling the laws of the block world
- Fitch is not tuned for arithmetic.
  Thus Fitch cannot handle the proof that $x \cdot x > x + x$

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- System F is not tuned for Tarski's world, and therefore has no special proof rules for the predicates there, like `LeftOf`
- But since the identity relation "=" is used in all domains of discourse, system F has special rules for that predicate

Identity Introduction (= Intro)

Rule described p.55 is amazingly simple:

$$\triangleright \mid t = t$$

This says that the identity relation is reflexive.

Properties of Identity

Symmetry

```
| a = b
|------
|
| b = a
```

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Symmetry

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|---
| b = a

Transitivity

| a = b
| b = c
|---
| a = c

Properties of Identity

Symmetry

```
| 1. a = b
|----
| 2. a = a    = Intro
| 3. b = a    = Elim: 2, 1
```

Transitivity

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| a = b
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Symmetry

1.	a = b	
2.	a = a	**= Intro**
3.	b = a	**= Elim: 2, 1**

Transitivity

1.	a = b	
2.	b = c	
3.	a = c	**= Elim: 1, 2**