Mechanics of Static Analysis

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Outline

1. Small-step semantics: trace generation
2. State generation and collecting semantics
3. Data-flow analysis
4. Ensuring termination
5. Typing rules and big-step semantics
6. Interprocedural analysis

Static analysis

A *static analysis* of a program is a *sound, finite, and approximate* calculation of the program's execution semantics.

Approximate: not exact — computes properties or aspects of the execution semantics, such as pre- or post-conditions, invariants, data types, patterns of trace, or ranges-of-values.

Sound: consistent with the concrete, execution semantics — a sound *overapproximation* describes a superset of the program's executions (**safe descriptions**); a sound *underapproximation* describes a subset of the program's executions (**live descriptions**). We will focus on overapproximations.

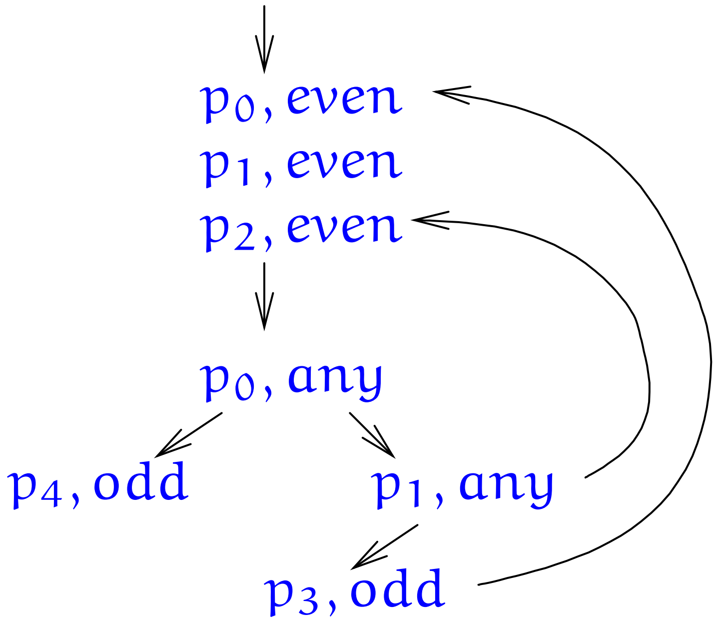
Finite: regardless of the program and its approximate semantics, the analysis terminates.

The most basic static analysis is *trace generation*

```
p0 : while (x != 1) {
   p1 : if Even(x)
      p2 : then  x = x div2;
      p3 : else  x = 3*x + 1;
     }
p4 :  exit
```

Note: p_i, v abbreviates $p_i, \langle x : v \rangle$

Abstract overapproximating trace:



$p_0, even$
$p_1, even$
$p_2, even$

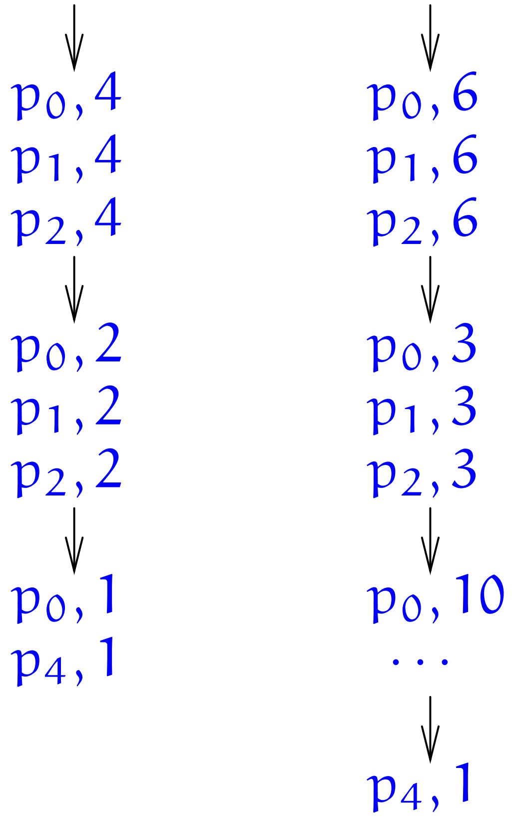
p_0, any

p_4, odd

p_1, any

p_3, odd

Two concrete traces:



$p_0, 4$
$p_1, 4$
$p_2, 4$

$p_0, 2$
$p_1, 2$
$p_2, 2$

$p_0, 1$
$p_4, 1$

$p_0, 6$
$p_1, 6$
$p_2, 6$

$p_0, 3$
$p_1, 3$
$p_2, 3$

$p_0, 10$
$\ldots$

$p_4, 1$

The abstract tree *(abstract model)* is a static analysis of those concrete executions that use an even-valued input.

Each concrete transition, $p_i, s \longrightarrow p_j, f_i(s)$, is reproduced by a corresponding abstract transition, $p_i, a \longrightarrow p_j, f_i^{\#}(a)$, where $s \in \gamma(a)$. ($f_i^{\#} = \alpha \circ f_i \circ \gamma$.)

The traces embedded in the abstract trace tree simulate all the concrete traces, e.g., this concrete trace,

$$p_0, 4 \longrightarrow p_1, 4 \longrightarrow p_2, 4 \longrightarrow p_0, 2 \longrightarrow p_1, 2 \longrightarrow p_2, 2 \longrightarrow p_0, 1 \longrightarrow p_4, 1$$

is simulated by this abstract trace, which is extracted from the abstract computation tree:

$$p_0, even \longrightarrow p_1, even \longrightarrow p_2, even \longrightarrow p_0, even \longrightarrow p_1, even \longrightarrow p_2, even \longrightarrow p_0, odd \longrightarrow p_4, odd$$

because we used a Galois connection to justify the soundness of the transition steps in the abstract trace tree.

In this fashion, a static analysis can generate an *abstract test or abstract model*, which covers a range of concrete inputs.

State reachability and collecting semantics

If we are interested *only in the reachable states and not their orderings in the trace*, we compute the program's *collecting semantics* as a nondecreasing sequence of sets of program states. The collecting semantics is an *abstraction* of trace-generation semantics.

Collecting semantics, concrete and abstract:

$\{p_0, 4\}$

$\{p_0, 4;\ p_1, 4\}$

$\{p_0, 4;\ p_1, 4;\ p_2, 4\}$

$\{p_0, 4;\ p_1, 4;\ p_2, 4;\ p_0, 2\}$

$\dots$

$\{p_0, 4;\ p_1, 4;\ p_2, 4;\ p_0, 2;$
$\quad p_1, 2;\ p_2, 2;\ p_0, 1;\ p_4, 1\}$

$\{p_0, even\}$

$\{p_0, even;\ p_4, even;\ p_1, even\}$

$\{p_0, even;\ p_4, even;\ p_1, even;\ p_2, even\}$

$\{p_0, even;\ p_4, even;\ p_1, even;\ p_2, even;$
$\quad p_0, any\}$

$\dots$

$\{p_0, even;\ p_4, even;\ p_1, even;\ p_2, even;$
$\quad p_0, any;\ p_4, any;\ p_1, any;\ p_3, odd\}$

“Sticky” collecting semantics

A semantics of form, $\wp(\mathrm{ProgramPoint} \times \mathrm{AbsStore})$, is “attaching” $\mathrm{AbsStore}$ values to each program point — the isomorphic representation, $\mathrm{ProgramPoint} \rightarrow \wp(\mathrm{AbsStore})$, is called the *(relational) “sticky” collecting semantics*:

$$\begin{aligned}&[p_0 \mapsto \{\mathrm{even}, \mathrm{any}\};\ p_1 \mapsto \{\mathrm{even}, \mathrm{any}\};\ p_2 \mapsto \{\mathrm{even}\};\\ &\quad p_3 \mapsto \{\mathrm{odd}\};\ p_4 \mapsto \{\mathrm{even}, \mathrm{any}\}]\end{aligned}$$

The above can be abstracted to a function in $\mathrm{ProgramPoint} \rightarrow \mathrm{AbsStore}$, the *independent-attribute* semantics:

$$[p_0 \mapsto \mathrm{any};\ p_1 \mapsto \mathrm{any};\ p_2 \mapsto \mathrm{even};\ p_3 \mapsto \mathrm{odd};\ p_4 \mapsto \mathrm{any}]$$

which is based on this abstraction mapping:

$$\alpha : \wp(\mathrm{AbsStore}) \rightarrow \mathrm{AbsStore}$$

$$\alpha(S) = \langle i : \textstyle\bigsqcup_{s \in S} s(i)\rangle_{i \in \mathrm{Identifier}}$$

Notice that the independent-attribute semantics is less precise than its relational ancestor; for example, variables `x` and `y` might have these values at program point p_i:

$$[...p_i \mapsto \{\langle x : even,\ y : even\rangle,\ \langle x : odd,\ y : odd\rangle\}...]$$

meaning that `x + y` computes to $even$ at p_i.

But the independent-attribute abstraction,

$$[...p_i \mapsto \langle x : any,\ y : any\rangle...]$$

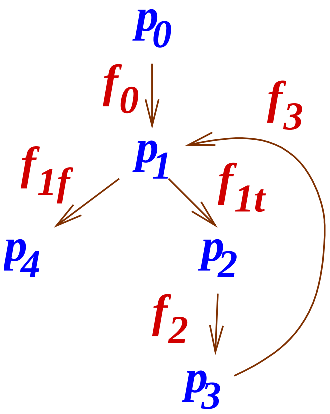
makes `x + y` compute to any, losing precision.

Note also that we could define a collecting version of a trace-generation semantics, which generates an analysis of form $ProgramPoint \to \wp(Trace)$.

Formalizing the “small steps”: *transfer functions*

A trace’s transitions, $pp_i, s \longrightarrow pp_i, s'$, are computed with a *control-flow graph* annotated with *transfer functions*.

```
p0 :  y = 1;
p1 :  while Even(x) {
         p2 : y = y * x;
         p3 : x = x div2;
      }
p4 :  exit
```



Concrete transfer functions: $\langle u, v\rangle$ abbreviates $\langle x : u, y : v\rangle$

$$f_0\langle u, v\rangle = \langle u, 1\rangle$$

$$f_{1t}(s) = \begin{cases} s & \text{if } s = \langle 2u, v\rangle \\ \perp & \text{otherwise} \end{cases} \qquad f_{1f}(s) = \begin{cases} s & \text{if } s = \langle 2u+1, v\rangle \\ \perp & \text{otherwise} \end{cases}$$

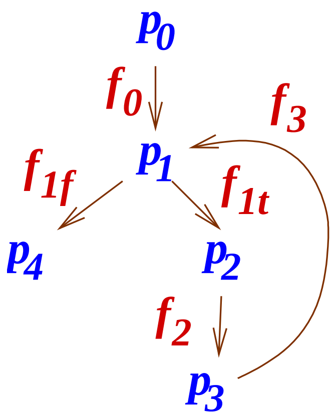
$$f_2\langle u, v\rangle = \langle u, v * u\rangle$$

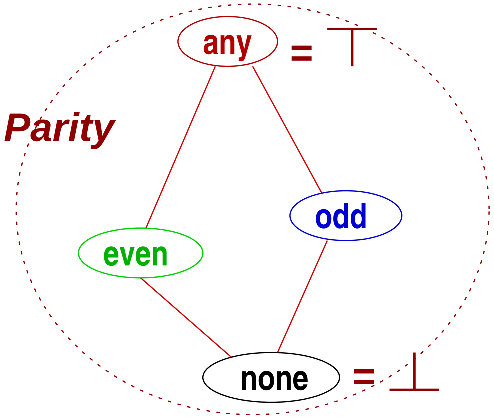
$$f_3\langle u, v\rangle = \langle u/2, v\rangle$$

Important: *configurations of form,* $p_i, \perp$, *cannot appear in a trace.*

The *abstract transfer functions* are derived as $f^{\#} = \alpha \circ f \circ \gamma$

```
p0 :  y = 1;
p1 :  while Even(x) {
         p2 : y = y * x;
         p3 : x = x div2;
      }
p4 :  exit
```





As usual, $\langle u, v\rangle$ abbreviates $\langle x : u, y : v\rangle$

Note: all $f^{\#}$ are *totally strict:* $f^{\#}\langle u, \bot\rangle = f^{\#}\langle \bot, v\rangle = \langle \bot, \bot\rangle$

$$f_0^{\#}\langle u, v\rangle = \langle u, \mathrm{odd}\rangle$$

$$f_{1t}^{\#} s = s \sqcap \langle \mathit{even}, \top\rangle \qquad f_{1f}^{\#} s = s \sqcap \langle \mathrm{odd}, \top\rangle$$

$$f_2^{\#}\langle u, v\rangle = \langle u, w\rangle, \text{ where } w = \begin{cases} \mathit{even} & \text{if } u = \mathit{even} \text{ or } v = \mathit{even}, \text{ else} \\ \mathrm{odd} & \text{if } u = \mathrm{odd} \text{ and } v = \mathrm{odd}, \text{ else} \\ \top & \end{cases}$$

$$f_3^{\#}\langle u, v\rangle = \langle \top, v\rangle$$

Note: $\langle a, b\rangle \sqcap \langle a', b'\rangle = \langle a \sqcap a', b \sqcap b'\rangle$.

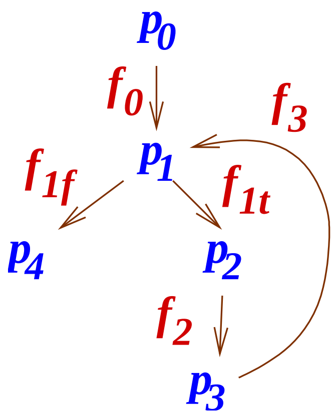
Flow equations calculate the (sticky, collecting) independent-attribute semantics

The value “attached” to program point p_i is defined by the equational pattern,

$$p_i\mathrm{Store} = \bigsqcup_{p_j \in pred(p_i)} f_j^{\#}(p_j\mathrm{Store})$$

The collecting semantics of p_i is the join of the answers computed by p_i’s predecessor transfer functions.

Flow equations for previous example:



$$p_0\mathrm{Store} = \langle x : \top, y : \top \rangle$$

$$p_1\mathrm{Store} = f_0^{\#}(p_0\mathrm{Store}) \sqcup f_3^{\#}(p_3\mathrm{Store})$$

$$p_2\mathrm{Store} = f_{1t}^{\#}(p_1\mathrm{Store})$$

$$p_3\mathrm{Store} = f_2^{\#}(p_2\mathrm{Store})$$

$$p_4\mathrm{Store} = f_{1f}^{\#}(p_1\mathrm{Store})$$

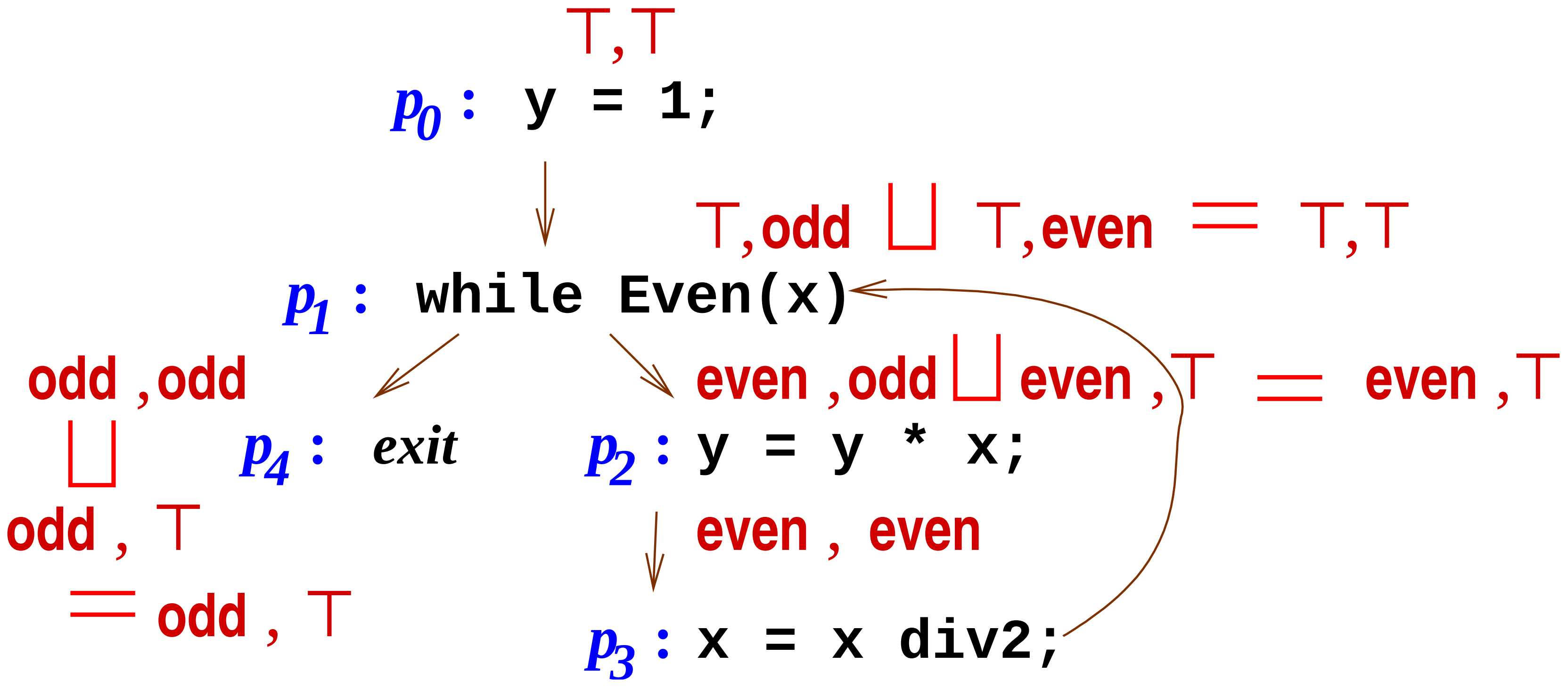
We *solve* the flow equations by calculating approximate solutions in stages until *the least fixed point* is reached.

Note: u, v abbreviates $\langle x : u, y : v \rangle$.

stage	$p_0\mathit{Store}$	$p_1\mathit{Store}$	$p_2\mathit{Store}$	$p_3\mathit{Store}$	$p_4\mathit{Store}$
0	$\bot, \bot$	$\bot, \bot$	$\bot, \bot$	$\bot, \bot$	$\bot, \bot$
1	$\top, \top$	$\bot, \bot$	$\bot, \bot$	$\bot, \bot$	$\bot, \bot$
2	$\top, \top$	$\top, \mathit{odd}$	$\bot, \bot$	$\bot, \bot$	$\bot, \bot$
3	$\top, \top$	$\top, \mathit{odd}$	$\mathit{even}, \mathit{odd}$	$\bot, \bot$	$\mathit{odd}, \mathit{odd}$
4	$\top, \top$	$\top, \mathit{odd}$	$\mathit{even}, \mathit{odd}$	$\mathit{even}, \mathit{even}$	$\mathit{odd}, \mathit{odd}$
$\ldots$					
8	$\top, \top$	$\top, \top$	$\mathit{even}, \top$	$\mathit{even}, \mathit{even}$	$\mathit{odd}, \top$
9	$\top, \top$	$\top, \top$	$\mathit{even}, \top$	$\mathit{even}, \mathit{even}$	$\mathit{odd}, \top$

A faster algorithm uses a *worklist* that remembers exactly which equations should be recalculated at each stage.

To summarize, we annotate the control-flow graph with the non-$\bot$ values that arrive at the program points:



$\top, \top$

p_0 : `y = 1;`

$\top, \mathsf{odd} \sqcup \top, \mathsf{even} = \top, \top$

p_1 : `while Even(x)`

$\mathsf{odd}, \mathsf{odd} \sqcup \mathsf{odd}, \top = \mathsf{odd}, \top$

p_4 : *exit*

$\mathsf{even}, \mathsf{odd} \sqcup \mathsf{even}, \top = \mathsf{even}, \top$

p_2 : `y = y * x;`

$\mathsf{even}, \mathsf{even}$

p_3 : `x = x div2;`

The analysis approximates the stores that arrive at the program points.

The equational format is called *data-flow analysis*. It is the most popular static analysis format.

Variants of data-flow analysis

We might vary whether the “data flow” goes forwards or backwards; we might also vary whether information is “joined” ($\sqcup$) or “met” ($\sqcap$):

Forwards-possibly:

$$p_i Store = \sqcup_{p_j \in pred(p_i)} f_j(p_j Store)$$

Forwards-necessarily:

$$p_i Store = \sqcap_{p_j \in pred(p_i)} f_j(p_j Store)$$

Backwards-possibly:

$$p_i Store = f_i^{-1}(\cup_{p_j \in succ(p_i)} p_j Store)$$

Backwards-necessarily:

$$p_i Store = f_i^{-1}(\cap_{p_j \in succ(p_i)} p_j Store)$$

The backwards analyses almost always compute sets of values, hence the use of $\cup$ and $\cap$.

A *forwards* analysis computes “histories” that arrive at a point:

forwards analysis = postcondition semantics

$p_i Store = a$ approximates the set of traces of the form
$p_0, s_0 \rightarrow p_1, s_1 \rightarrow \cdots \rightarrow p_i, s_i$ (where $s_i \in \gamma(a)$)

A *backwards* analysis computes the “futures” from a program point:

backwards analysis = precondition semantics

$p_i Store = a$ approximates the set of traces of the form
$p_i, s_i \rightarrow \cdots \rightarrow p_{exit}, s_{final}$ (where $s_i \in \gamma(a)$)

A *possibly* analysis predicts a “superset” of the actual computations: if $p_i Store = a$, then *for all* concrete values, $c \sqsubseteq_C \gamma(a)$, that arrive at p_i, we have $c \sqsubseteq_C \gamma(a)$ — all possibilities are predicted.

A *necessarily* analysis predicts a “subset” of the actual computations: if $p_i Store = a$, then *there exists* some $c \sqsubseteq_C \gamma(a)$, that arrives at p_i.

The data-flow example developed earlier in this Lecture computed answers of the form,

$$p_i\mathrm{Store} = a$$

which asserted, if store s arrives at program point p_i, then $s \in \gamma(a)$.

But there are data-flow analyses where $p_i\mathrm{Store} = a$ means that all *execution traces* that arrive at p_i contain some *pattern* of program points and stores, described by a.

We will develop the Galois-connection formalities in the next Lecture, but just now we study two examples, used by compilers for improving register allocation in target code. *These examples compute sets of program phrases that describe patterns within execution traces.*

The examples show variations of the forwards/backwards and possibly/necessarily forms of data-flow analysis.

Forwards-necessarily-reaching definitions: which assignments *must* reach their successors

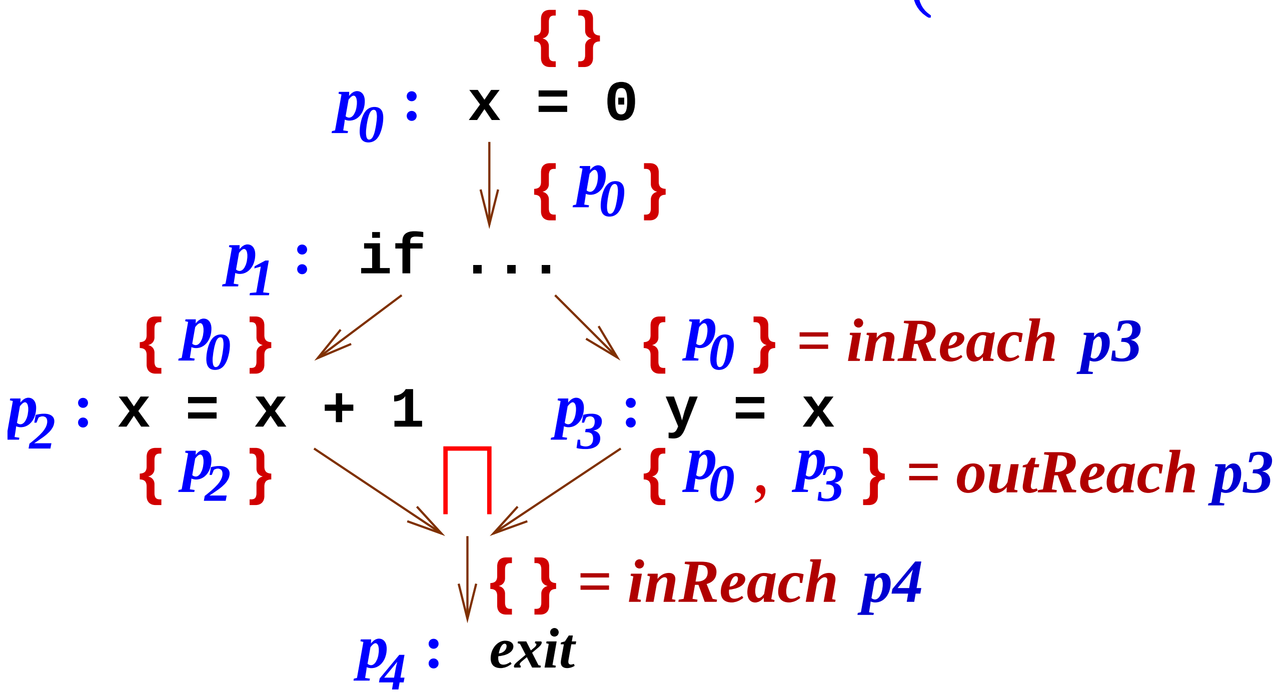
$$\mathrm{inReach}\,p_i = \bigcap_{p_j \in \mathrm{pred}(p_i)} \mathrm{outReach}\,p_j$$

$$\mathrm{outReach}\,p_i = f_i^{\#}(\mathrm{inReach}\,p_i) = (\mathrm{inReach}\,p_i - \mathrm{kill}_i) \cup \mathrm{gen}_i$$

(the transfer function computes a set of assignment statements)

for $p_i : \mathtt{x} = \mathtt{e}, \begin{cases} \mathrm{kill}_i = \{p_j \mid p_j : \mathtt{x} = \ldots\} \\ \mathrm{gen}_i = \{p_i\} \end{cases}$ for $p_i : \mathtt{if}\ \mathtt{e}, \begin{cases} \mathrm{kill}_i = \{\} \\ \mathrm{gen}_i = \{\} \end{cases}$

Sample analysis:



$\{\ \}$

p_0 : `x = 0`

$\{ p_0 \}$

p_1 : `if ...`

$\{ p_0 \}$

$\{ p_0 \}$ = *inReach* $p3$

p_2 : `x = x + 1`

p_3 : `y = x`

$\{ p_2 \}$

$\sqcap$

$\{ p_0 , p_3 \}$ = *outReach* $p3$

$\{\ \}$ = *inReach* $p4$

p_4 : *exit*

Explanation:

If $p' \in inReach p_i$, where p' labels the assignment, $p' : v = e$, then *all* traces from p_0 to p_i *must* possess the pattern,

$$p_0 \rightarrow \cdots \rightarrow p' \rightarrow \cdots \rightarrow p_i$$

and no assignment, `v = e'`, occurs between p' and p_i in the trace.

If $p' \in inReach p_i$ holds, then the assignment at p' should save its right-hand-side value in a register for quick access by p_i.

Backwards-possibly-live variables: which variables *might* be referenced in the future

$$outLive p_i = \bigcup_{p_j \in succ(p_i)} inLive p_j$$

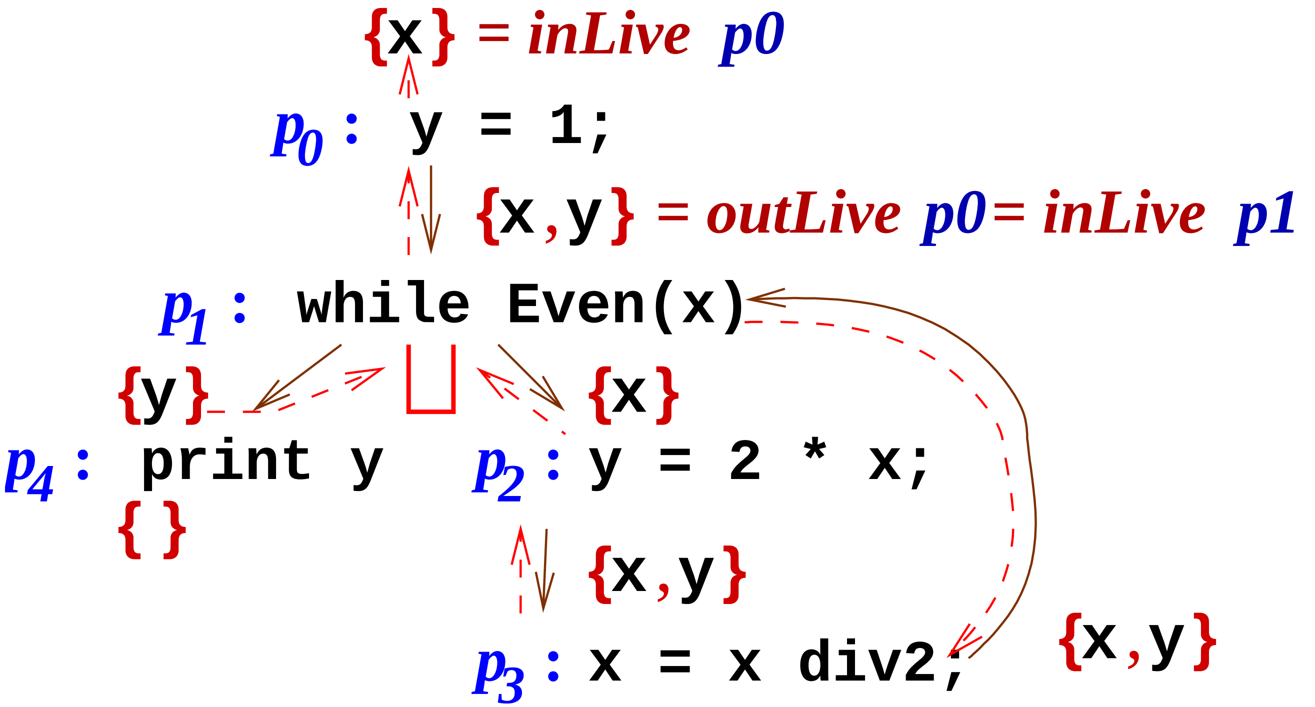
$$inLive p_i = f_i^{\#}(outLive p_i) = (outLive p_i - kill_i) \cup gen_i$$

(the transfer function computes a set of variable names)

for p_i : `x = e` $\begin{cases} kill_i = \{x\} \\ gen_i = \{v \mid v \text{ in } \mathtt{e}\} \end{cases}$

for `print e` / p_i : `while e` $\begin{cases} kill_i = \{\} \\ gen_i = \{v \mid v \text{ in } \mathtt{e}\} \end{cases}$

Sample analysis:



{x} = *inLive p0*

p_0 : `y = 1;`

{x,y} = *outLive p0* = *inLive p1*

p_1 : `while Even(x)`

$\bigsqcup$

{y}

p_4 : `print y`

{ }

{x}

p_2 : `y = 2 * x;`

{x,y}

p_3 : `x = x div2;`

{x,y}

Explanation:

If there is a concrete execution trace containing the pattern,

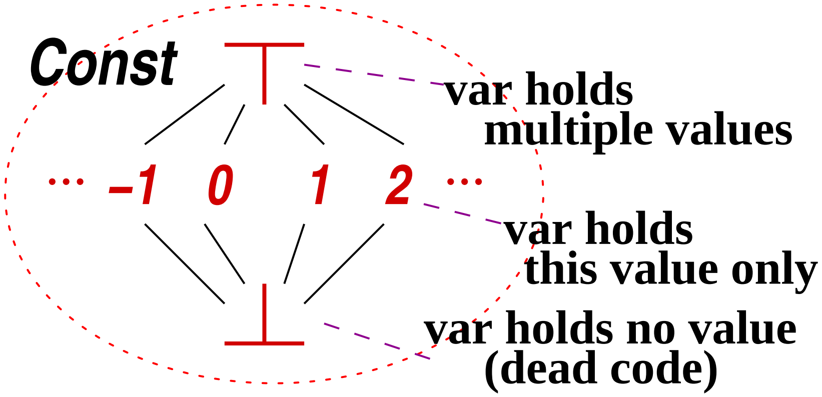
$$p_i \rightarrow \cdots \rightarrow p' \rightarrow \cdots \rightarrow p_{exit}$$

such that p' references variable v and no assignment to v appears between p_i and p', then $v \in outLive p_i$.

If $v \notin outLive p_i$ holds, then v's value should be removed from all registers upon completion of p_i's execution — v is a *“dead variable”* after p_i.

Termination: Constant propagation reviewed

```
p0 :  x = 1; y = 2;
p1 :  while (x < y + z)
        p2 :  x = x + 1;
      }
p3 :  exit
```



var holds multiple values

var holds this value only

var holds no value (dead code)

where $m + n$ is interpreted

$$k_1 + k_2 \longrightarrow sum(k_1, k_2),$$

$$\top \neq k_i \neq \bot, i \in 1..2$$

$$\top + k \longrightarrow \top$$

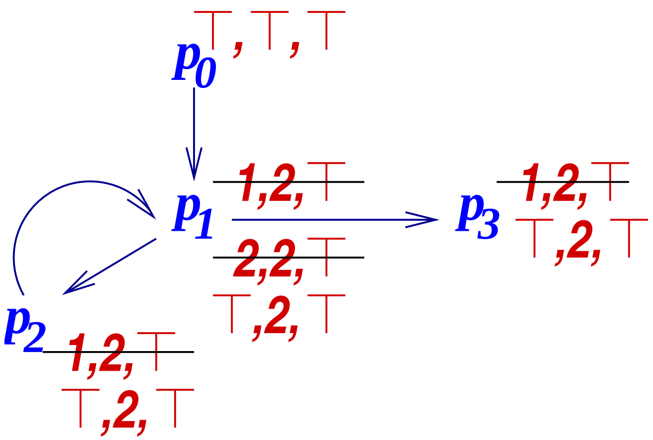
$$k + \top \longrightarrow \top$$

Abstract trace:

$$\begin{tikzcd}
p_0, \langle \top, \top, \top \rangle & \\
p_1, \langle 1, 2, \top \rangle \arrow[dd] \arrow[dr] & \\
& p_3, \langle 1, 2, \top \rangle \\
p_2, \langle 1, 2, \top \rangle & \\
p_1, \langle 2, 2, \top \rangle \arrow[dd] \arrow[dr] & \\
& p_3, \langle 2, 2, \top \rangle \\
p_2, \langle 2, 2, \top \rangle & \\
p_1, \langle 3, 2, \top \rangle \arrow[dd] \arrow[dr] & \\
& \ldots \\
\ldots &
\end{tikzcd}$$

The naive trace does not terminate.

Finite-height and $\sqcup$ give termination



stage	p_0Store	p_1Store	p_2Store	p_3Store
1	$\top, \top, \top$	$\bot, \bot, \bot$	$\bot, \bot, \bot$	$\bot, \bot, \bot$
2	$\top, \top, \top$	$1, 2, \top$	$\bot, \bot, \bot$	$\bot, \bot, \bot$
3	$\top, \top, \top$	$1, 2, \top$	$1, 2, \top$	$1, 2, \top$
4	$\top, \top, \top$	$\top, 2, \top$	$1, 2, \top$	$1, 2, \top$
5	$\top, \top, \top$	$\top, 2, \top$	$\top, 2, \top$	$\top, 2, \top$
6	$\top, \top, \top$	$\top, 2, \top$	$\top, 2, \top$	$\top, 2, \top$

Termination is *guaranteed* because the transfer functions and $\sqcup$ are monotonic (each stage has values not smaller than its predecessors) and the abstract domain, **Const**, has *finite height* — there are no infinitely ascending sequences (the stages cannot increase forever).

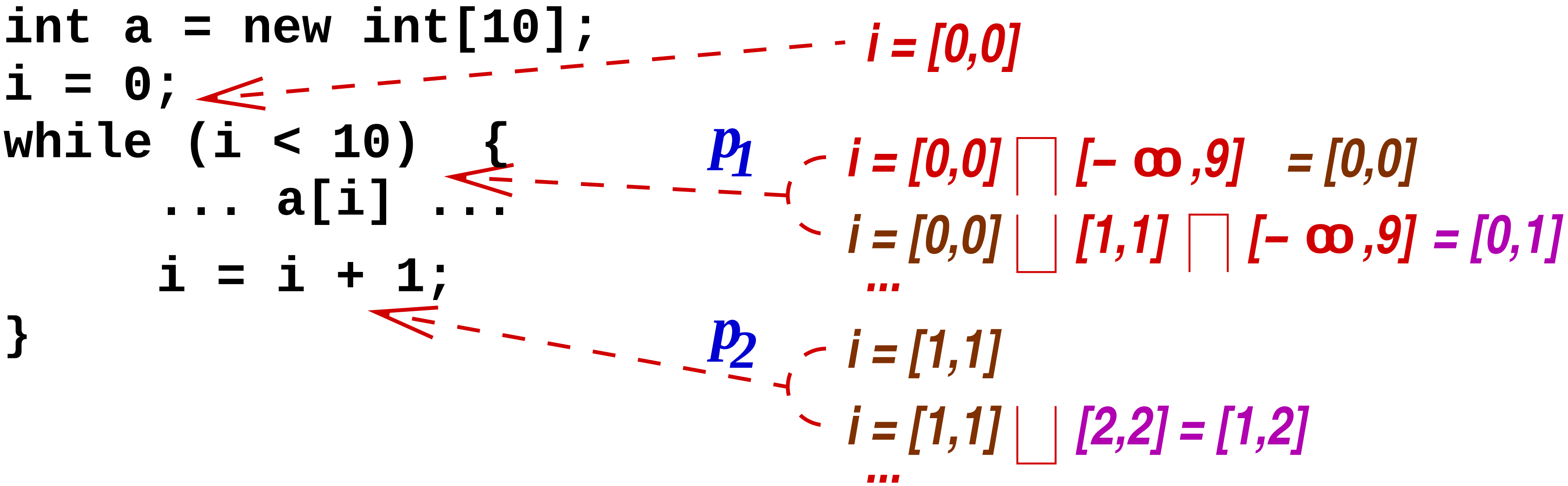
(Indeed, the longest seqence in **Const** goes: $\bot \sqsubseteq k \sqsubseteq \top$.)

Termination: Array-bounds checking reviewed

Integer variables receive values from the *interval domain*,

$$I = \{[i,j] \mid i,j \in \mathrm{Int} \cup \{-\infty, +\infty\}\}.$$

We define $[a,b] \sqcup [a',b'] = [\min(a,a'), \max(b,b')]$.



```
int a = new int[10];
i = 0;
while (i < 10)  {
     ... a[i] ...
     i = i + 1;
}
```

$i = [0,0]$

p_1

$i = [0,0] \sqcap [-\infty,9] = [0,0]$

$i = [0,0] \sqcup [1,1] \sqcap [-\infty,9] = [0,1]$

...

p_2

$i = [1,1]$

$i = [1,1] \sqcup [2,2] = [1,2]$

...

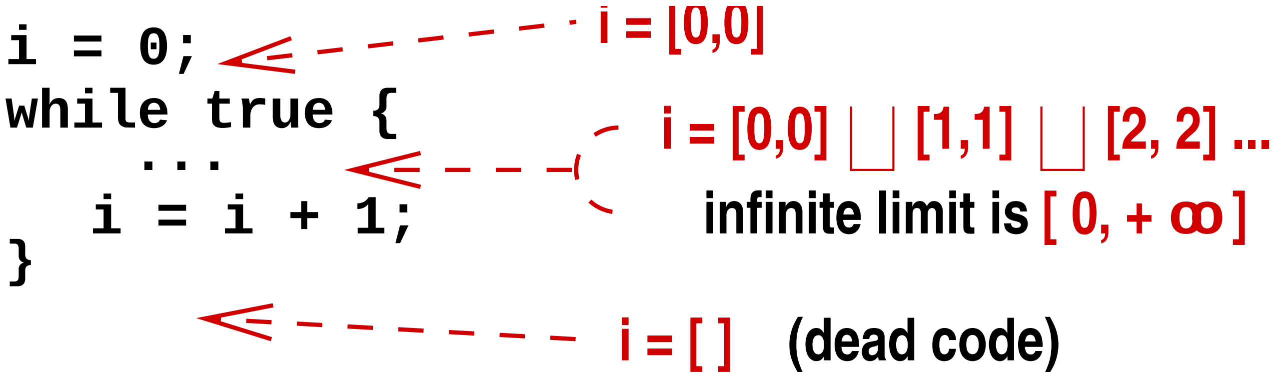
This example terminates: `i`'s ranges are

at $p_1 : [0..9]$

at $p_2 : [1..10]$

at loop exit $: [1..10] \sqcap [10,+\infty] = [10,10]$

But others might not, because the domain is not finite height:



```
i = 0;
while true {
    ...
  i = i + 1;
}
```

i = [0,0]

i = [0,0] $\sqcup$ [1,1] $\sqcup$ [2, 2] ...

infinite limit is [0, + ∞]

i = [ ]   (dead code)

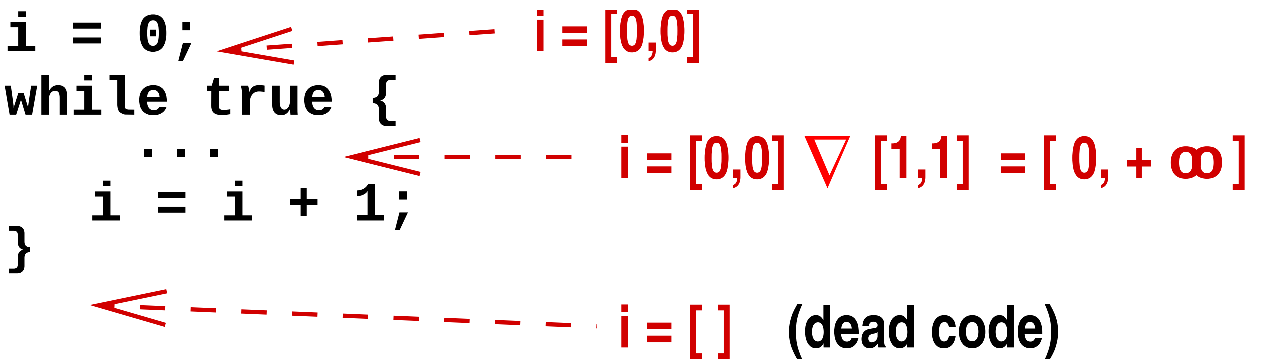
The analysis generates the infinite sequence of stages, $[0,0], [0,1], \ldots, [0,i], \ldots$ as `i`'s value in the loop's body.

The domain of intervals, where $[i,j] \sqsubseteq [i',j']$ *iff* $i \leq j$ *and* $j \leq j'$*, has infinitely ascending chains.*

To forcefully terminate the analysis, we can replace the $\sqcup$ operation by ∇, called a *widening operator*:

$$[\,]\nabla[i,j] = [i,j] \qquad [i,j]\nabla[i',j'] = \begin{array}{l}[\text{if } i' < i \text{ then } -\infty \text{ else } i,\\ \text{if } j' > j \text{ then } +\infty \text{ else } j]\end{array}$$

The widening operator, which guarantees finite convergence for all increasing sequences on the interval domain, quickly terminates the example:



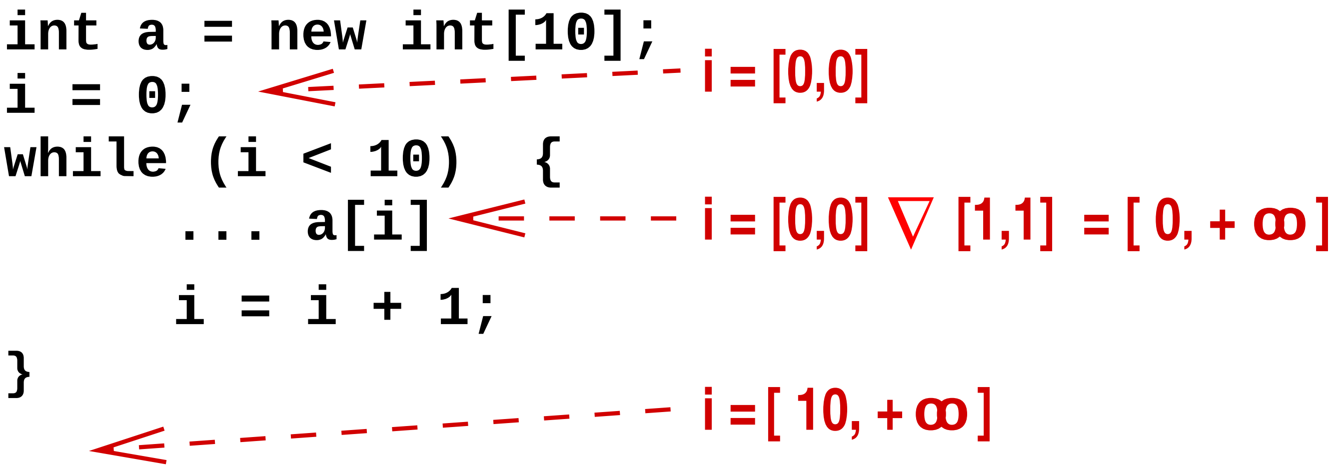
```
i = 0;
while true {
    ...
  i = i + 1;
}
```

$i = [0,0]$

$i = [0,0] \nabla [1,1] = [0, +\infty]$

$i = [\,]$ **(dead code)**

but in general, it can lose much precision:



```
int a = new int[10];
i = 0;
while (i < 10)  {
     ... a[i]
     i = i + 1;
}
```

$i = [0,0]$

$i = [0,0] \nabla [1,1] = [0, +\infty]$

$i = [10, +\infty]$

For this reason, a complementary operation, $\triangle$, called a *narrowing operation*, can be used after ∇ gives convergence to recover some precision and retain a fixed-point solution.

We will not develop $\triangle$ here, but for the interval domain, a suitable $\triangle$ tries to reduce $-\infty$ and $+\infty$ to finite values. For the last example, the convergent value, $[0, +\infty]$, in the loop body would be narrowed to $[0, 10]$, making `i`'s value on loop exit $[10, 10]$.

Another approach is to use multiple "thresholds" for widening, e.g. $-\infty$, $(2^{-31} - 1)$, 0, etc. for lower limits, and $(2^{31} - 1)$ and $+\infty$ for upper limits.

Structured (*big-step*) static analysis

Given a block of statements, B, we might wish to calculate the values that "enter" and "exit" from B. If B is coded in a structured language, we can define the static analysis to compute a *structured transfer function* for B:

$$\mathrm{C} \ ::= \ \mathrm{p} : \mathtt{x} \ = \mathrm{E} \ \mid \ \mathrm{C}_1; \mathrm{C}_2 \ \mid \ \mathtt{if}\ \mathrm{E}\ \mathrm{C}_1\ \mathrm{C}_2 \ \mid \ \mathtt{while}\ \mathrm{E}\ \mathrm{C}$$

A sample structured analysis that ignores tests: $[\![\mathrm{C}]\!] : A_{in} \rightarrow A_{out}$

$[\![\mathrm{p} : \mathtt{x} \ = \mathrm{E}]\!]in = f_p^{\#}(in)$ (the transfer function for p)

$[\![\mathrm{C}_1; \mathrm{C}_2]\!]in = [\![\mathrm{C}_2]\!]([\![\mathrm{C}_1]\!]in)$

$[\![\mathtt{if}\ \mathrm{E}\ \mathrm{C}_1\ \mathrm{C}_2]\!]in = [\![\mathrm{C}_1]\!]in \sqcup [\![\mathrm{C}_2]\!]in$

$[\![\mathtt{while}\ \mathrm{E}\ \mathrm{C}]\!]in = in \sqcup out_C,$

where $out_C = \bigsqcup_{i \geq 0} out_i,$

and $out_0 = \bot_A$ and $out_{i+1} = [\![\mathrm{C}]\!](in \sqcup out_i)$

We can annotate a syntax tree with the in-and out-data — here is a *forwards-possibly* reaching definitions analysis, which computes sets of assignments that might reach future program points:

$$\llbracket p : \texttt{x} \ = E \rrbracket \mathit{in} = \mathit{in} - \mathit{kill}_{\texttt{x}} \cup \{p\}$$

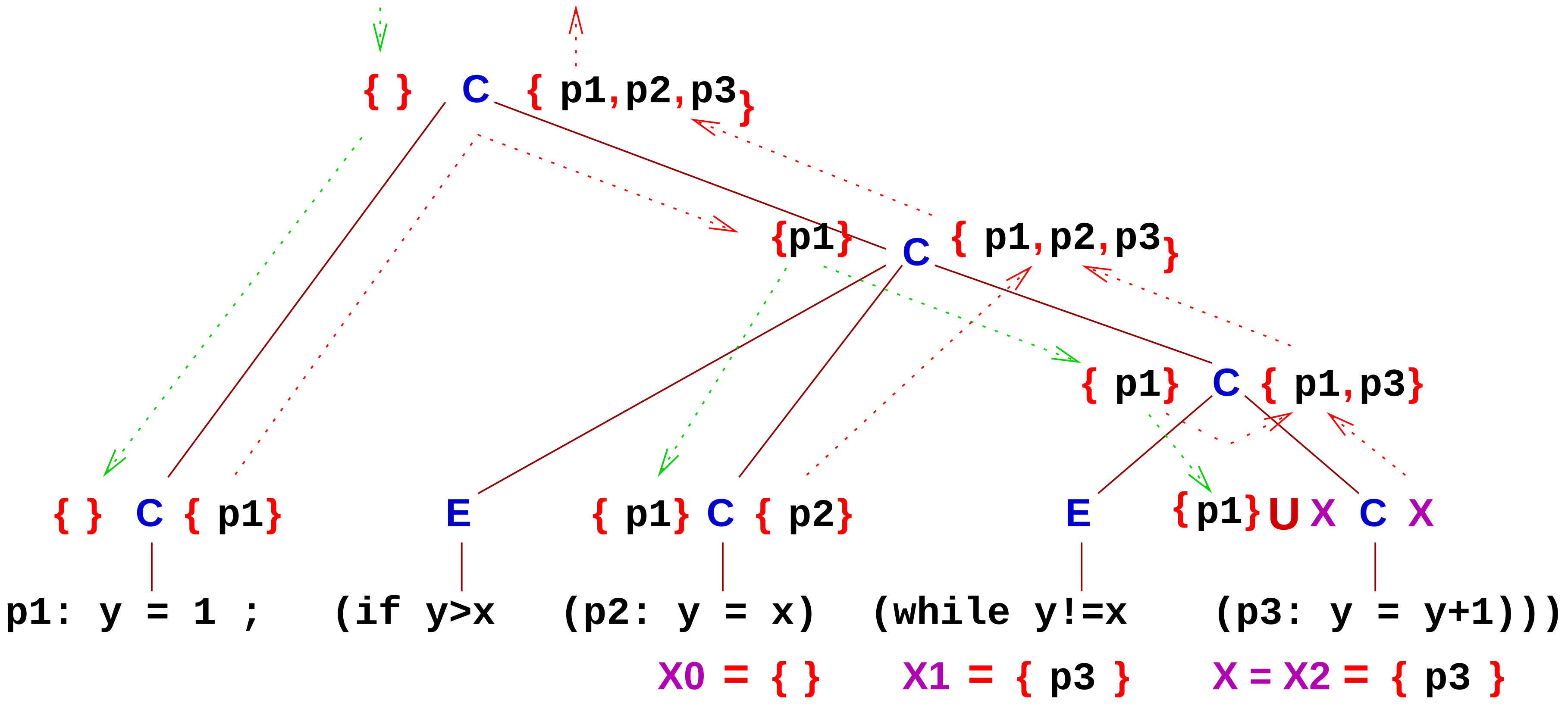
$$\llbracket C_1 ; C_2 \rrbracket \mathit{in} = \llbracket C_2 \rrbracket (\llbracket C_1 \rrbracket \mathit{in})$$

$$\llbracket \texttt{if}\ E\ C_1\ C_2 \rrbracket \mathit{in} = \llbracket C_1 \rrbracket \mathit{in} \ \cup \ \llbracket C_2 \rrbracket \mathit{in}$$

$$\llbracket \texttt{while}\ E\ C \rrbracket \mathit{in} = \mathit{in} \cup \bigcup_{i \geq 0} \mathit{out}_i,$$

where $\mathit{out}_0 = \{\}$

and $\mathit{out}_{i+1} = \llbracket C \rrbracket (\mathit{in} \cup \mathit{out}_i)$



The analysis calculates a “local” least-fixed point at each while-loop, in contrast to a data-flow analysis, which calculates a single “global” least-fixed point for the entire program. (It is straightforward to prove that both techniques compute the same answer.)

The structured, equational style is based on *denotational semantics*.

The structured analysis is no more precise than the iterative, data-flow analysis (that is, a sticky, collecting semantics); indeed, $[\![\mathrm{C}]\!] : A_{\mathrm{in}} \rightarrow A_{\mathrm{out}}$ is an abstraction of the data-flow analysis of C in the sense that $[\![\mathrm{C}]\!]$ “forgets” the flow information of C’s subphrases and returns only C’s output.

Structured analysis in inference-rule format

The style of the previous example suggests that a structured analysis pairs each phrase, C, with its input and outputs, in_c and out_c.

We might write relational “assertions” in the formats

$$in_c\ C\ out_c \qquad \text{or} \qquad C : in_c \to out_c.$$

The first format is used in Hoare-logics, the second in data-typing.

The semantics equations inspire us to write these inference rules:

$$\vdash p : \mathtt{x} = E : in \to f_p^{\#}(in) \qquad \frac{\vdash C_1 : in \to out \quad \vdash C_2 : out \to out'}{\vdash C_1; C_2 : in \to out'}$$

$$\frac{\vdash C_1 : in \to out_1 \quad \vdash C_2 : in \to out_2}{\vdash \mathtt{if}\ E\ C_1\ C_2 : in \to out_1 \sqcup out_2} \qquad \frac{\vdash C : in \sqcup out \to out}{\vdash \mathtt{while}\ E\ C : in \to out}$$

We use the rules to derive a program’s analysis as a “proof.”

Reaching definitions, repeated:

$$\mathrm{in}\ p : \texttt{x} \ = E\ \mathrm{in} - \mathrm{kill}_x \cup \{p\} \qquad \frac{\mathrm{in}\ C_1\ \mathrm{out} \quad \mathrm{out}\ C_2\ \mathrm{out}'}{\mathrm{in}\ C_1; C_2\ \mathrm{out}'}$$

$$\frac{\mathrm{in}\ C_1\ \mathrm{out}_1 \quad \mathrm{in}\ C_2\ \mathrm{out}_2}{\mathrm{in}\ \texttt{if}\ E\ C_1\ C_2\ \mathrm{out}_1 \cup \mathrm{out}_2} \qquad \frac{\mathrm{in} \cup \mathrm{out}\ C\ \mathrm{out}}{\mathrm{in}\ \texttt{while}\ E\ C\ \mathrm{out}}$$

The (inverted) proof resembles the annotated syntax tree:

$$\dfrac{\{\,\}\ \texttt{p1: y = 1 ; if y>x ...}\ \{\,\texttt{p1,p2,p3}\}}{\{\,\}\ \texttt{p1: y = 1}\ \{\,\texttt{p1}\} \qquad \dfrac{\{\,\texttt{p1}\}\ \texttt{if y>x ...}\ \{\,\texttt{p1,p2,p3}\}}{\{\,\texttt{p1}\}\ \texttt{p2: y = x}\ \{\,\texttt{p2}\} \qquad \dfrac{\{\,\texttt{p1}\}\ \texttt{while y!=x ...}\ \{\,\texttt{p1,p3}\}}{\{\,\texttt{p1,p3}\}\ \texttt{p3: y = y+1}\ \{\,\texttt{p3}\}}}}$$

Unlike the denotational-semantics version, the `while`-rule does *not* calculate a least-fixed point: A “guess” or “inference” of an *invariant assertion* is made to obtain a proof of $\mathrm{in} \cup \mathrm{out} C \mathrm{out}$ that is used to prove $\mathrm{in}\texttt{while}\ E\ C \mathrm{out}$. *The program analysis is done in one pass*.

The example suggests that data-typing systems defined as "inference-rule sets" are *one-pass, structured static analyses*:

$$\pi \vdash x : \pi(x) \qquad \frac{\pi \oplus [x \mapsto \tau_1] \vdash E : \tau_2}{\pi \vdash \lambda x.E : \tau_1 \rightarrow \tau_2} \qquad \frac{\pi \vdash E_1 : \tau_1 \rightarrow \tau_2 \quad \pi \vdash E_2 : \tau_1}{\pi \vdash E_1\ E_2 : \tau_2}$$

These data-typing rules, which underlie the ML languages, analyze a program in one pass and predict the range of values *(data type)* that the program's phrases will produce when executed:

$$\gamma(\mathit{bool}) = \mathit{Bool} = \{\mathit{true}, \mathit{false}\}$$
$$\gamma(\mathit{int}) = \mathit{Int} = \{\ldots -1, 0, 1, \ldots\}$$
$$\gamma(\tau_1 \rightarrow \tau_2) = \{f : \mathit{Val} \mid \text{for all } a \in \gamma(\tau_1),\ f(a) \in \gamma(\tau_2)\}$$

where $\mathit{Val} = \bigcup_{i \geq 0} V_i$, such that $V_0 = \{\}$ and $V_{i+1} = \mathit{Bool} \cup \mathit{Int} \cup (V_i \rightarrow V_i)$.

A guess is needed for τ_1 in the hypotheis of the second typing rule. A typical implementation of the rules uses first-order unification to calculate an intelligent guess.

Big-step (natural) semantics is a multi-pass analysis

$$\sigma \vdash p : x \;=\; E \Downarrow f_p(\sigma)$$

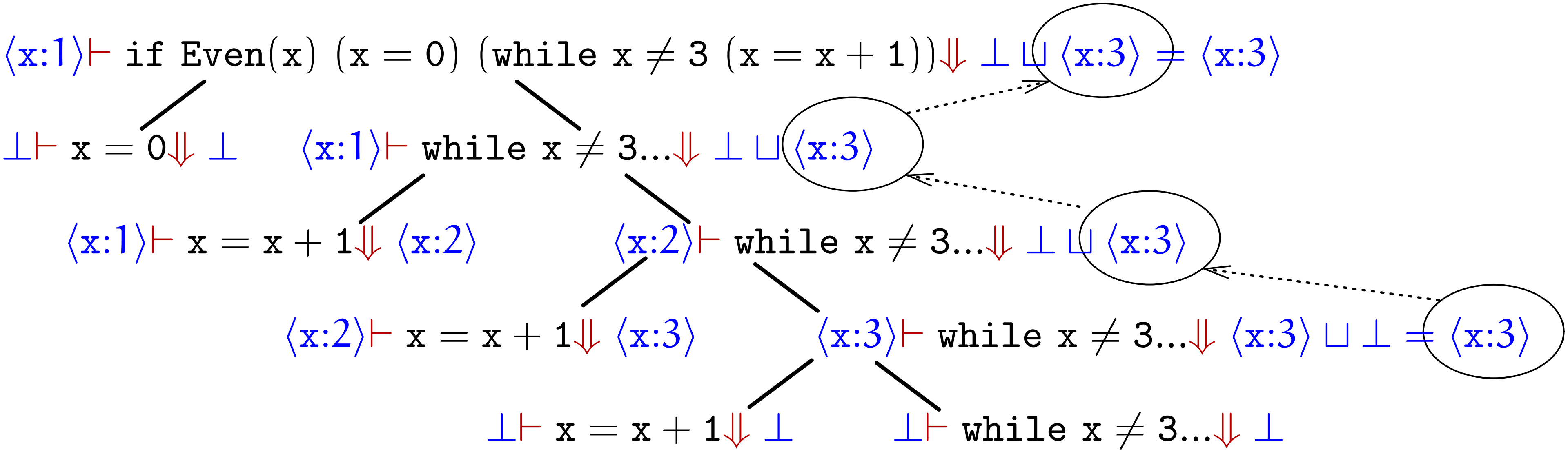
$$\frac{\sigma \vdash C_1 \Downarrow \sigma_1 \quad \sigma_1 \vdash C_2 \Downarrow \sigma_2}{\sigma \vdash C_1; C_2 \Downarrow \sigma_2} \qquad \frac{f_{Et}(\sigma) \vdash C_1 \Downarrow \sigma_1 \quad f_{Ef}(\sigma) \vdash C_2 \Downarrow \sigma_2}{\sigma \vdash \texttt{if}\ E\ C_1\ C_2 \Downarrow \sigma_1 \sqcup \sigma_2}$$

$$\frac{f_{Et}(\sigma) \vdash C \Downarrow \sigma' \quad \sigma' \vdash \texttt{while}\ E\ C \Downarrow \sigma''}{\sigma \vdash \texttt{while}\ E\ C \Downarrow f_{Ef}(\sigma) \sqcup \sigma''} \qquad \bot \vdash C \Downarrow \bot$$

Recall that f_p is a transfer function and that f_{Et} and f_{Ef} ”filter” the store, e.g.,

$f_{x>2t}\langle x:4, y:3\rangle = \langle x:4, y:3\rangle$, whereas $f_{x>2t}\langle x:0, y:3\rangle = \bot$.

An example: `if Even(x) (x=0) (while x≠3 (x = x+1))`



$\langle x{:}1\rangle \vdash \texttt{if Even}(x)\ (x = 0)\ (\texttt{while}\ x \neq 3\ (x = x + 1)) \Downarrow \bot \sqcup \langle x{:}3\rangle = \langle x{:}3\rangle$

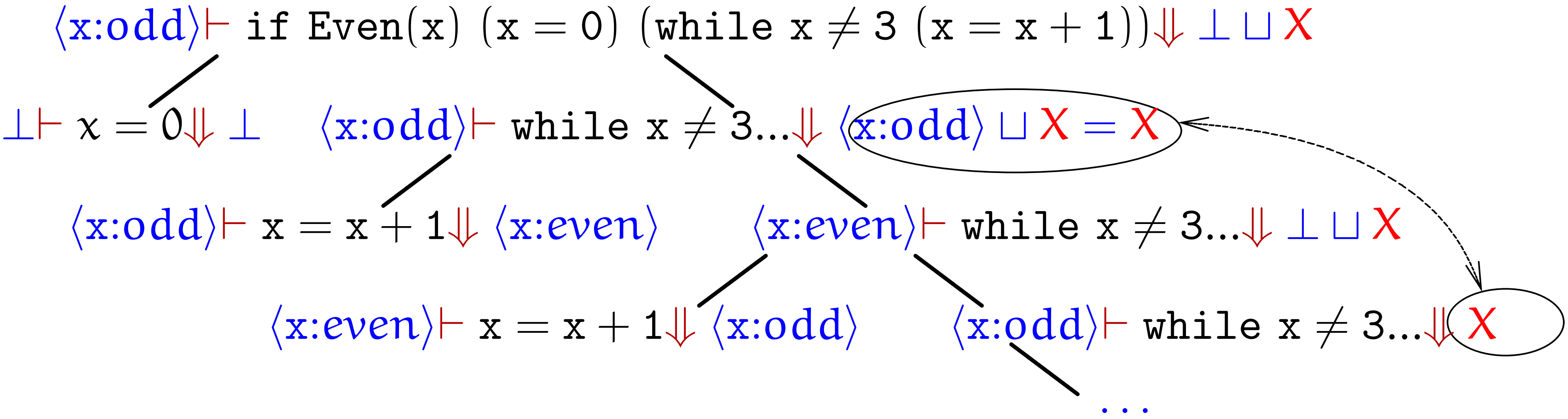
$\bot \vdash x = 0 \Downarrow \bot$ $\quad$ $\langle x{:}1\rangle \vdash \texttt{while}\ x \neq 3 \ldots \Downarrow \bot \sqcup \langle x{:}3\rangle$

$\langle x{:}1\rangle \vdash x = x + 1 \Downarrow \langle x{:}2\rangle$ $\quad$ $\langle x{:}2\rangle \vdash \texttt{while}\ x \neq 3 \ldots \Downarrow \bot \sqcup \langle x{:}3\rangle$

$\langle x{:}2\rangle \vdash x = x + 1 \Downarrow \langle x{:}3\rangle$ $\quad$ $\langle x{:}3\rangle \vdash \texttt{while}\ x \neq 3 \ldots \Downarrow \langle x{:}3\rangle \sqcup \bot = \langle x{:}3\rangle$

$\bot \vdash x = x + 1 \Downarrow \bot$ $\quad$ $\bot \vdash \texttt{while}\ x \neq 3 \ldots \Downarrow \bot$

An abstract big-step tree: using the same inference rules but with abstract transfer functions for **Parity** $= \{\bot, \mathit{even}, \mathrm{odd}, \top\}$, we generate an abstract tree that is *infinite* but *regular*:



$$\langle \mathrm{x{:}odd}\rangle \vdash \texttt{if Even(x) (x} = \texttt{0) (while x} \neq \texttt{3 (x} = \texttt{x} + \texttt{1))} \Downarrow \bot \sqcup X$$

$$\bot \vdash \texttt{x} = \texttt{0} \Downarrow \bot \qquad \langle \mathrm{x{:}odd}\rangle \vdash \texttt{while x} \neq \texttt{3}\ldots \Downarrow \langle \mathrm{x{:}odd}\rangle \sqcup X = X$$

$$\langle \mathrm{x{:}odd}\rangle \vdash \texttt{x} = \texttt{x} + \texttt{1} \Downarrow \langle \mathrm{x{:}}\mathit{even}\rangle \qquad \langle \mathrm{x{:}}\mathit{even}\rangle \vdash \texttt{while x} \neq \texttt{3}\ldots \Downarrow \bot \sqcup X$$

$$\langle \mathrm{x{:}}\mathit{even}\rangle \vdash \texttt{x} = \texttt{x} + \texttt{1} \Downarrow \langle \mathrm{x{:}odd}\rangle \qquad \langle \mathrm{x{:}odd}\rangle \vdash \texttt{while x} \neq \texttt{3}\ldots \Downarrow X$$

$$\ldots$$

Variable X denotes the answer from the repeated loop subderivation:

$$X = \langle \mathrm{x{:}odd}\rangle \sqcup X$$

The least solution sets $X = \langle \mathrm{x{:}odd}\rangle$.

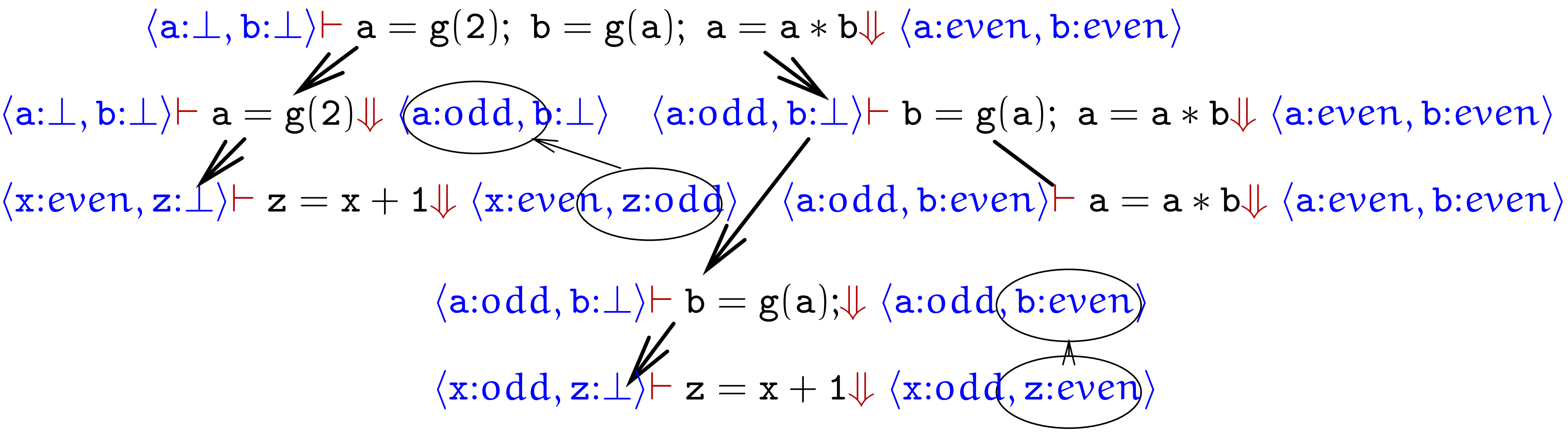
Big-step semantics naturally supports interprocedural analysis

$$\frac{\texttt{func f(x) local y; } C. \qquad [x \mapsto [\![E]\!]\sigma][y \mapsto \bot] \vdash C \Downarrow \sigma'}{\sigma \vdash z \ = \ f(E) \Downarrow \sigma[z \mapsto \sigma'(y)]}$$

where $[\![E]\!]\sigma$ denotes E's value with σ, and $x \mapsto v$ assigns v to x.

Example:

```
func g(x) local z; z = x+1.
a = g(2); b = g(a); a = a*b
```



$\langle a{:}\bot, b{:}\bot\rangle \vdash a = g(2);\ b = g(a);\ a = a * b \Downarrow \langle a{:}even, b{:}even\rangle$

$\langle a{:}\bot, b{:}\bot\rangle \vdash a = g(2) \Downarrow \langle a{:}odd, b{:}\bot\rangle$

$\langle a{:}odd, b{:}\bot\rangle \vdash b = g(a);\ a = a * b \Downarrow \langle a{:}even, b{:}even\rangle$

$\langle x{:}even, z{:}\bot\rangle \vdash z = x + 1 \Downarrow \langle x{:}even, z{:}odd\rangle$

$\langle a{:}odd, b{:}even\rangle \vdash a = a * b \Downarrow \langle a{:}even, b{:}even\rangle$

$\langle a{:}odd, b{:}\bot\rangle \vdash b = g(a); \Downarrow \langle a{:}odd, b{:}even\rangle$

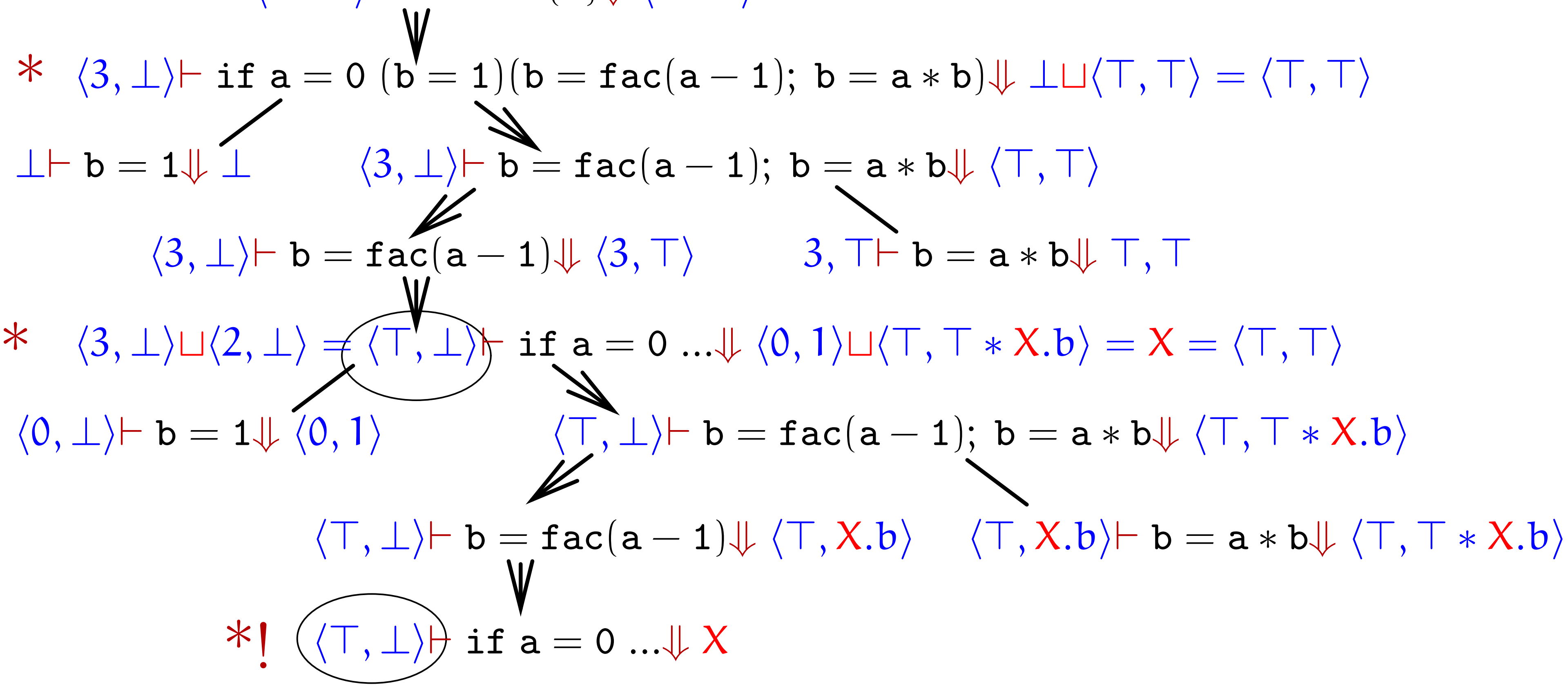
$\langle x{:}odd, z{:}\bot\rangle \vdash z = x + 1 \Downarrow \langle x{:}odd, z{:}even\rangle$

The derivation tree naturally separates the calling contexts.

Recursions (*) force accelerated termination (!):

$\mathtt{func\ fac(a)\ local\ b;\ if\ a = 0\ (b\ =\ 1)\ (b\ =\ fac(a-1);\ b\ =\ a\ *\ b)}$

$\mathtt{c\ =\ fac(3)}$



$\langle c:\bot\rangle\vdash \mathtt{c = fac(3)}\Downarrow \langle c:\top\rangle$

* $\langle 3,\bot\rangle\vdash \mathtt{if\ a = 0\ (b = 1)(b = fac(a-1);\ b = a*b)}\Downarrow \bot\sqcup\langle\top,\top\rangle = \langle\top,\top\rangle$

$\bot\vdash \mathtt{b = 1}\Downarrow \bot$

$\langle 3,\bot\rangle\vdash \mathtt{b = fac(a-1);\ b = a*b}\Downarrow \langle\top,\top\rangle$

$\langle 3,\bot\rangle\vdash \mathtt{b = fac(a-1)}\Downarrow \langle 3,\top\rangle$

$3,\top\vdash \mathtt{b = a*b}\Downarrow \top,\top$

* $\langle 3,\bot\rangle\sqcup\langle 2,\bot\rangle = \boxed{\langle\top,\bot\rangle}\vdash \mathtt{if\ a = 0\ ...}\Downarrow \langle 0,1\rangle\sqcup\langle\top,\top*X.b\rangle = X = \langle\top,\top\rangle$

$\langle 0,\bot\rangle\vdash \mathtt{b = 1}\Downarrow \langle 0,1\rangle$

$\langle\top,\bot\rangle\vdash \mathtt{b = fac(a-1);\ b = a*b}\Downarrow \langle\top,\top*X.b\rangle$

$\langle\top,\bot\rangle\vdash \mathtt{b = fac(a-1)}\Downarrow \langle\top,X.b\rangle$

$\langle\top,X.b\rangle\vdash \mathtt{b = a*b}\Downarrow \langle\top,\top*X.b\rangle$

*! $\boxed{\langle\top,\bot\rangle}\vdash \mathtt{if\ a = 0\ ...}\Downarrow X$

$X = \langle 0,1\rangle\sqcup\langle\top,\top*X.b\rangle$ **The least solution sets** $X = \langle\top,\top\rangle$.

The traditional data-flow implementation uses *call strings*:

- Each procedure has its own control-flow graph, as does the main program. Each procedure invocation and return is drawn as a “goto” arc in the graph for the entire program.
- When the program is analyzed, the store is accompanied by a calling history, called the *call string*. (E.g., `main` calls `p` — the call string is `"main::p"`.)
- A finite bound, k, is placed on the call string’s length — only the the k most recent invocations are remembered.
- Say that the call string is S, execution is in `p`, and `p` calls `q`. The call string is revised to $S' = (S :: p) \downarrow k$, and `q`’s activation record labelled S' is used to execute `q`. At conclusion, control returns to $S.\mathit{last}$ — `p` — and the call string is shortened. (If the call string is empty, then the return “gos to” all possible return points!)

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