# Introduction to Abstraction and Static Analysis

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## Outline

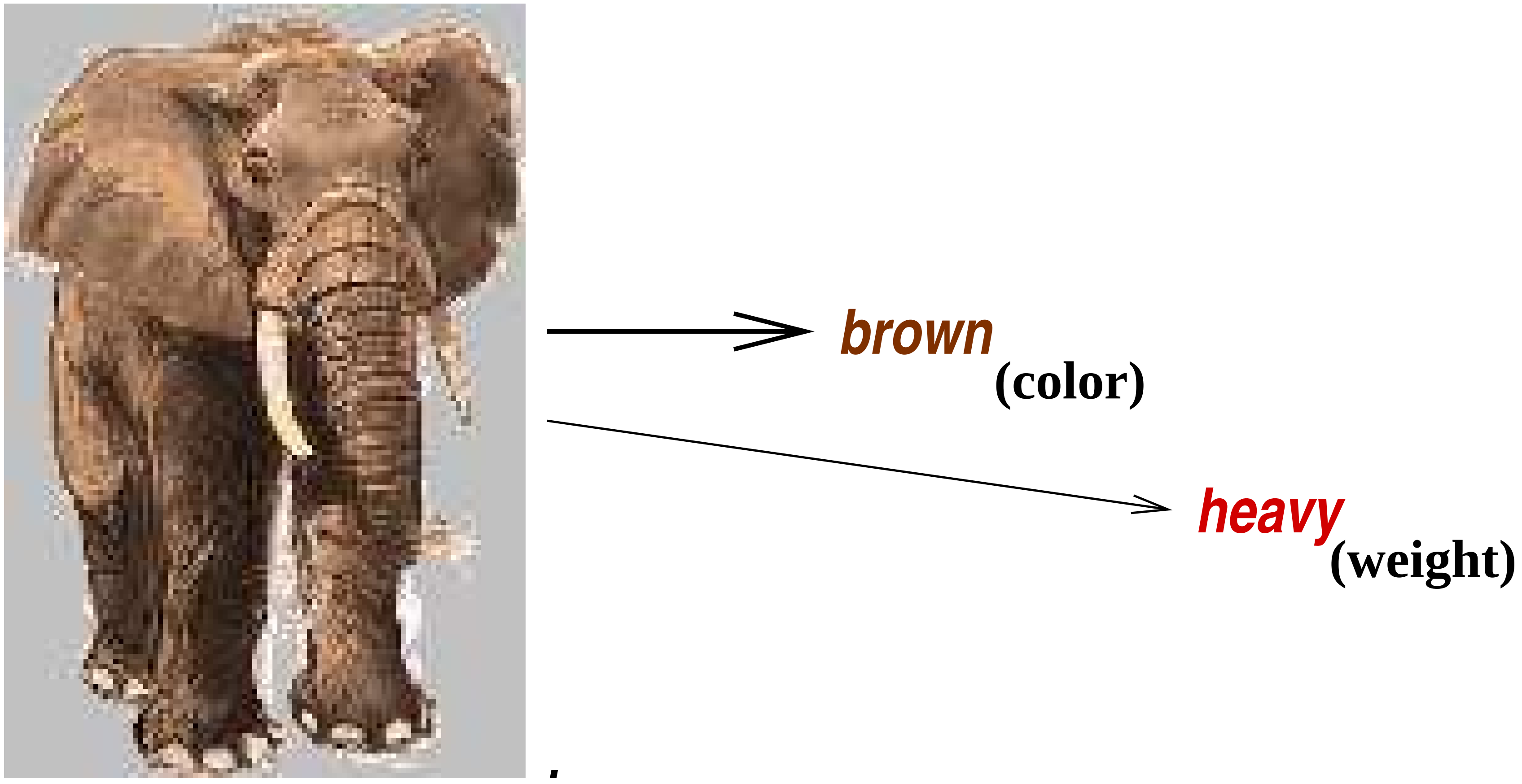
1. What is abstraction?
2. Abstraction and concretization:
   Galois-connection-based abstract interpretation
3. Examples of static analyses
4. Logics and static analysis

## An *abstraction* is a property from some domain

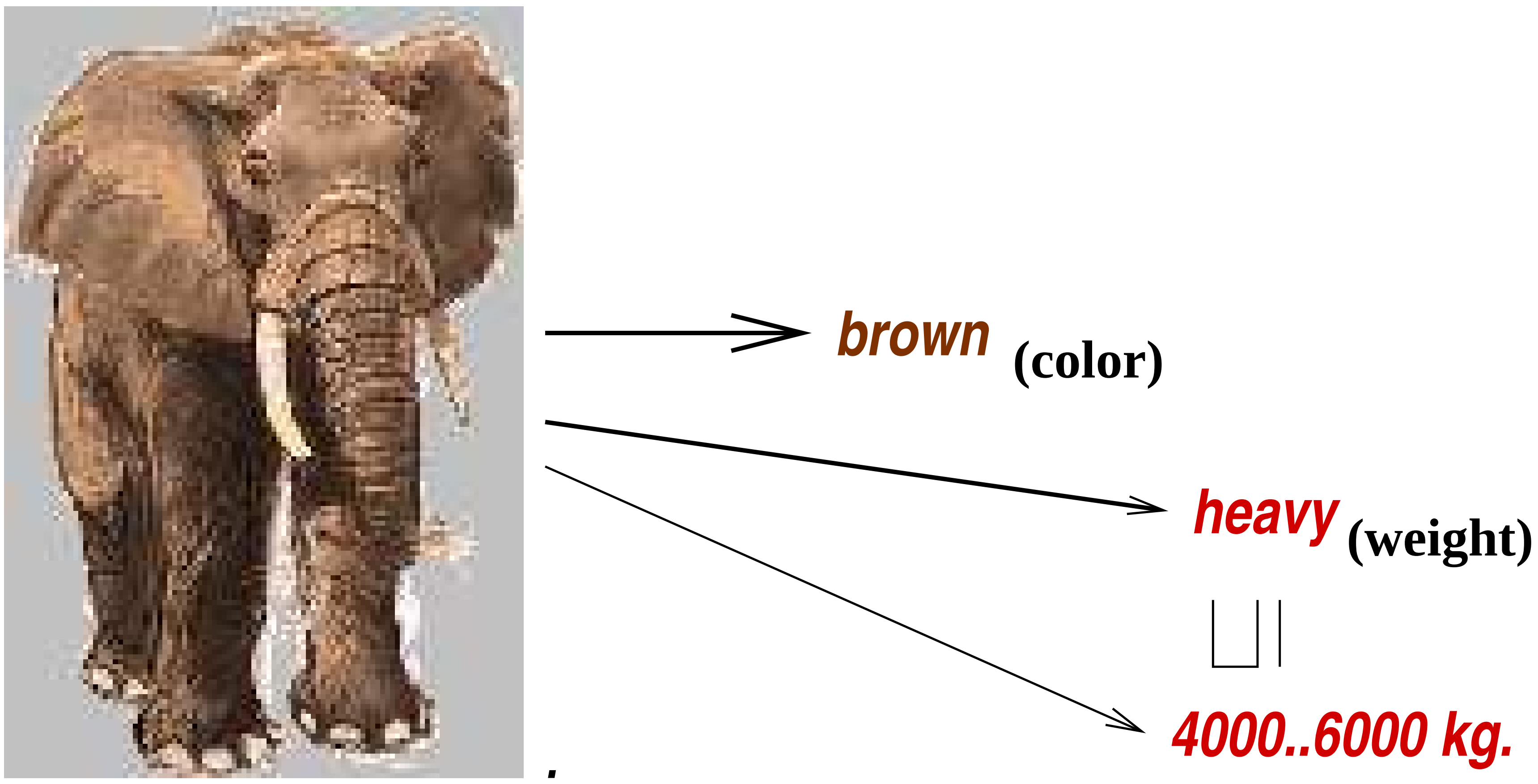
$\longrightarrow$ *brown* (color)

.

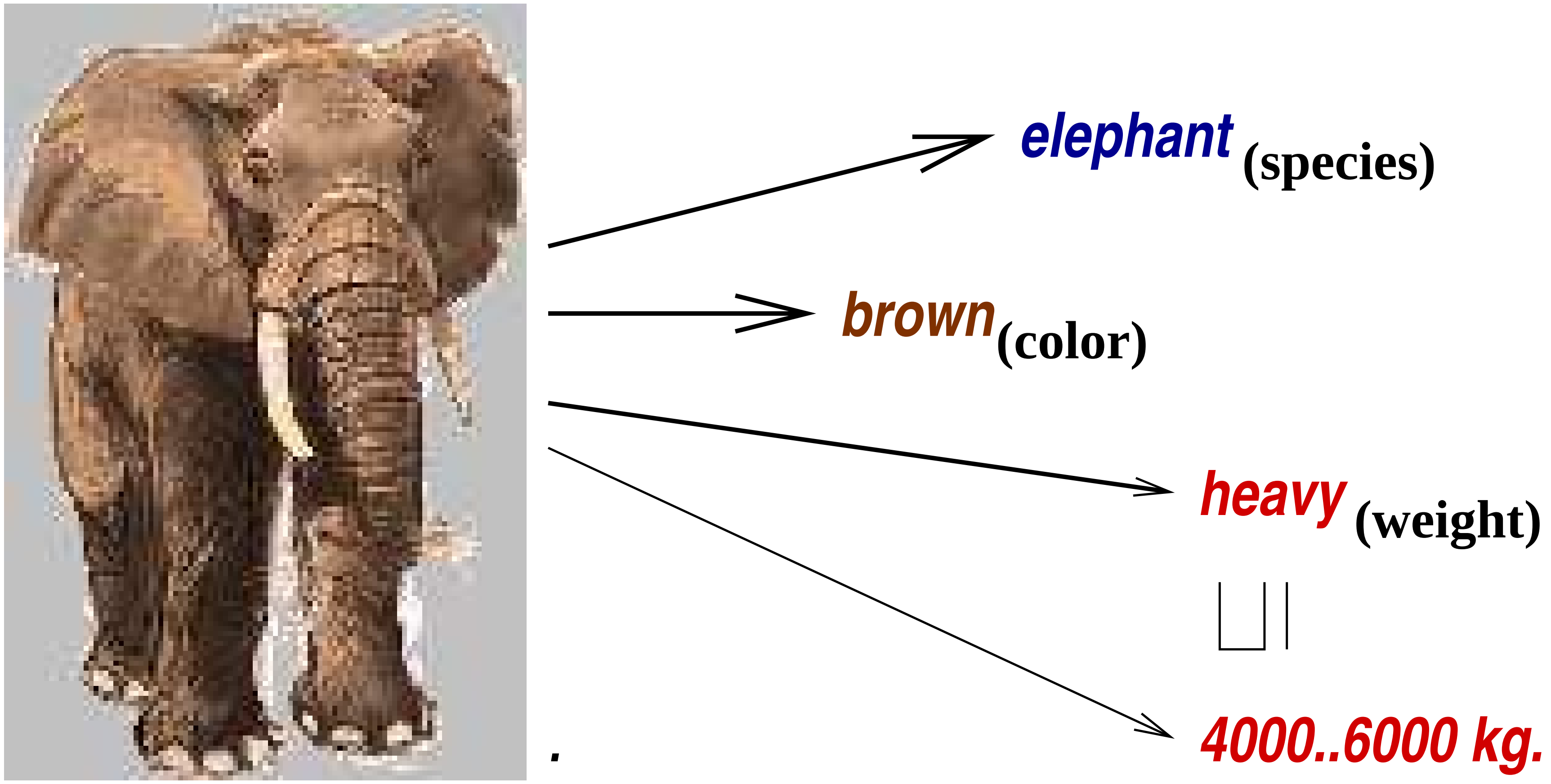
## An *abstraction* is a property (cont.)



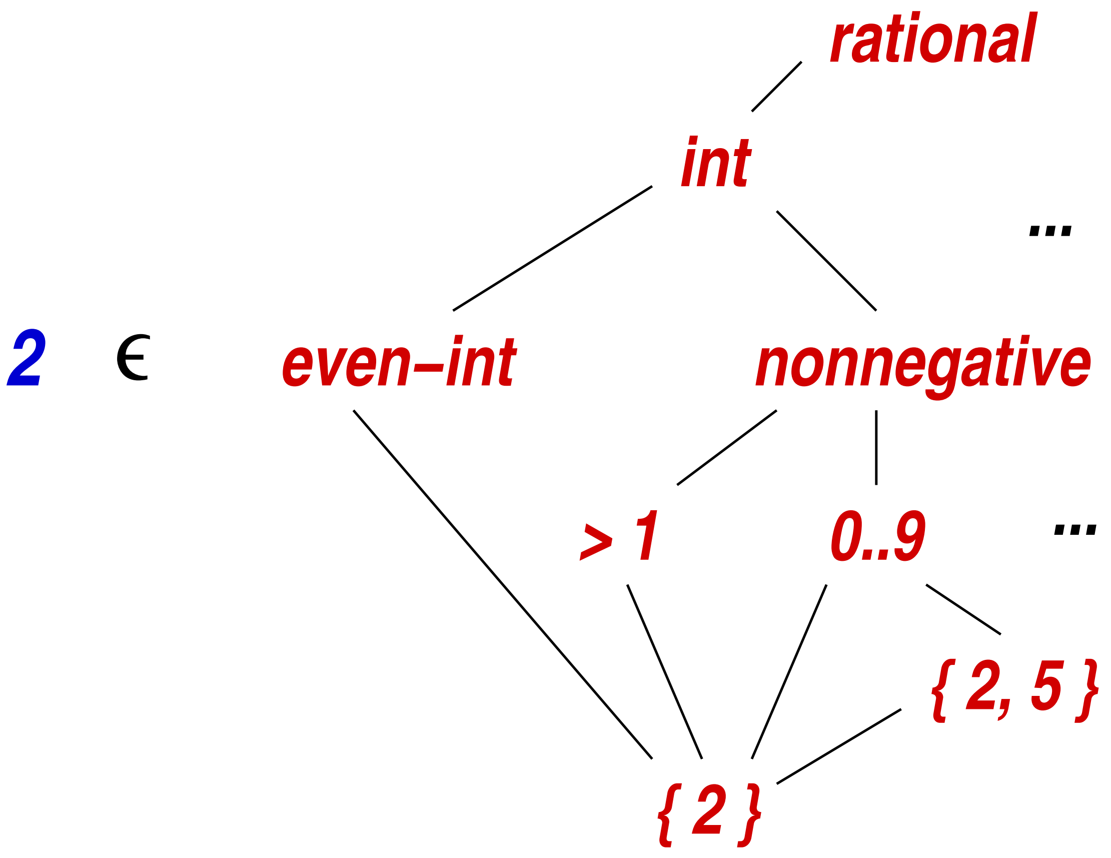
## An *abstraction* is a property (cont.)



## An *abstraction* is a property (concl.)

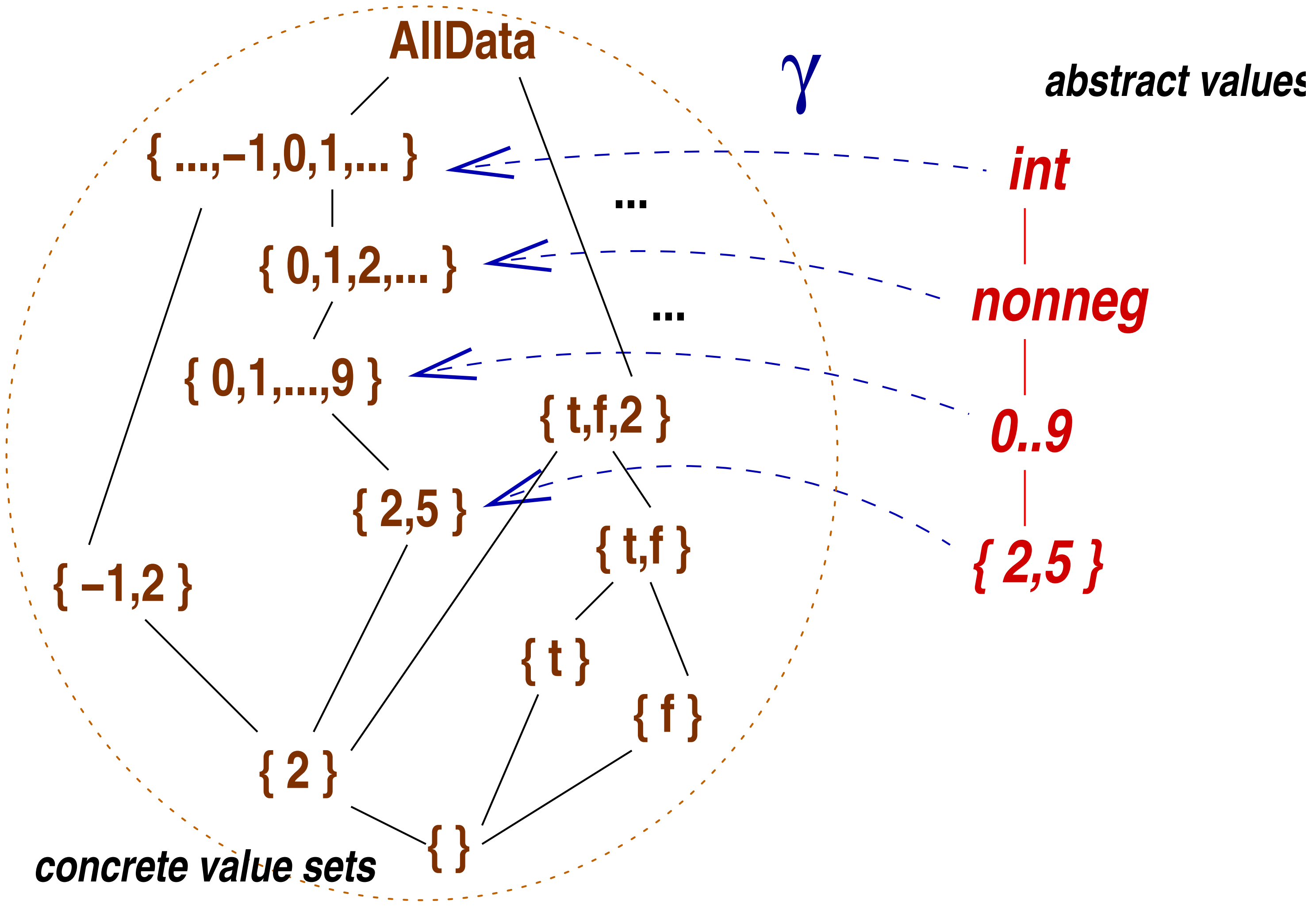


## *Value abstractions* are classic to computing



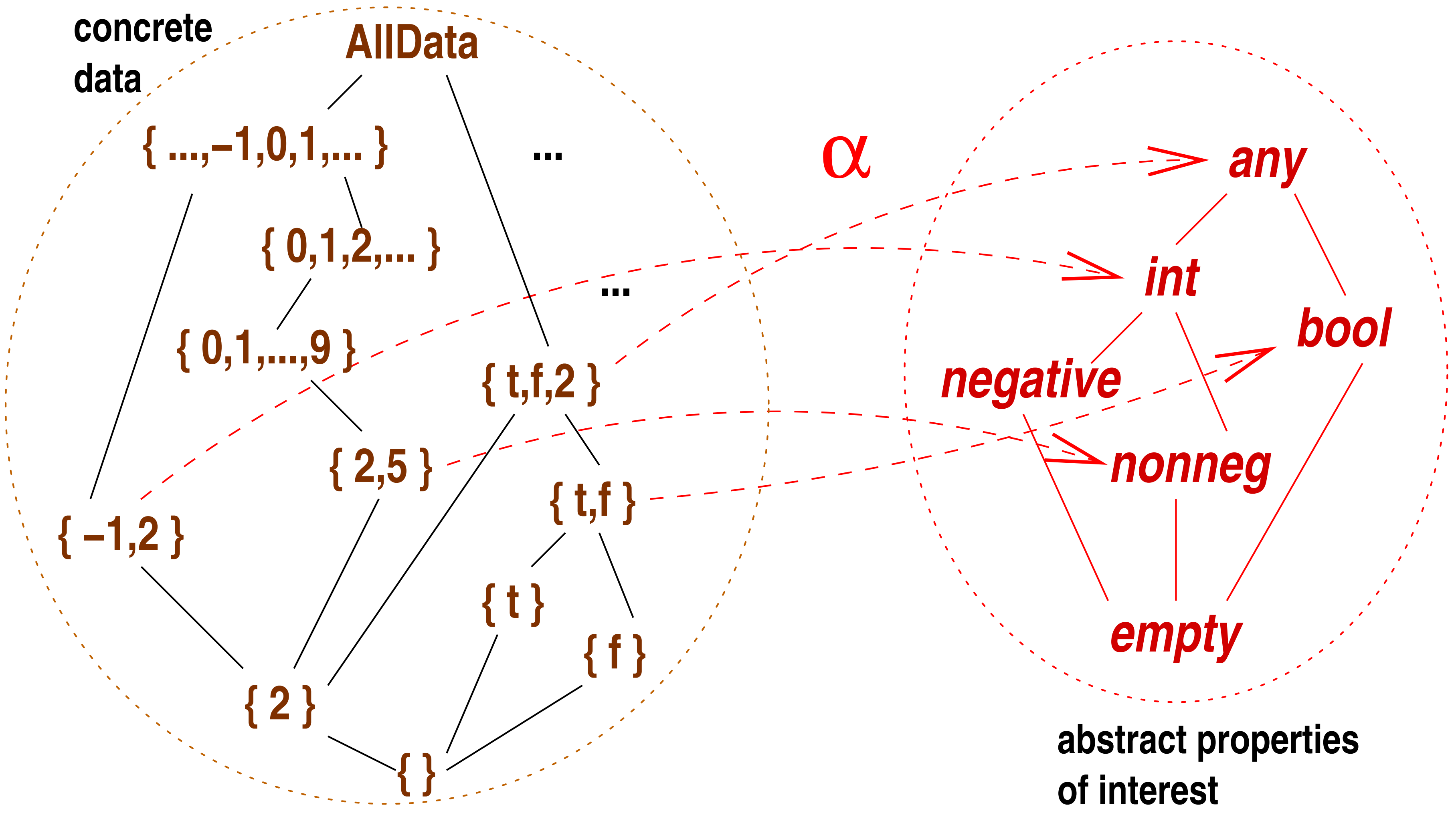
All the properties listed on the right are abstractions of $2$; the upwards lines denote $\sqsubseteq$, a loss of precision.

## Abstract values name sets of concrete values



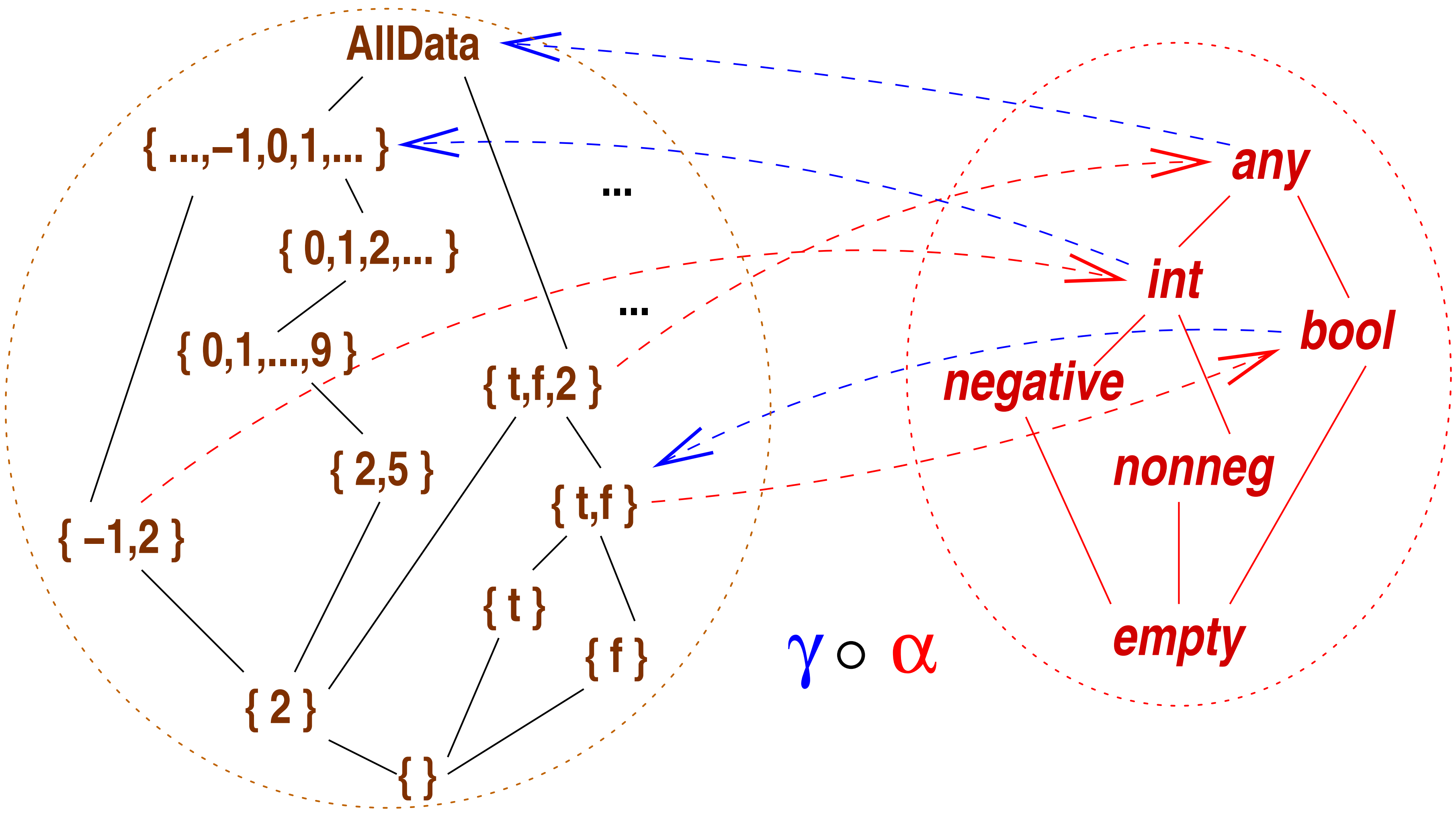
Function $\gamma$ maps each abstract value to the set of concrete values it represents.

## Sets of concrete values are abstracted imprecisely



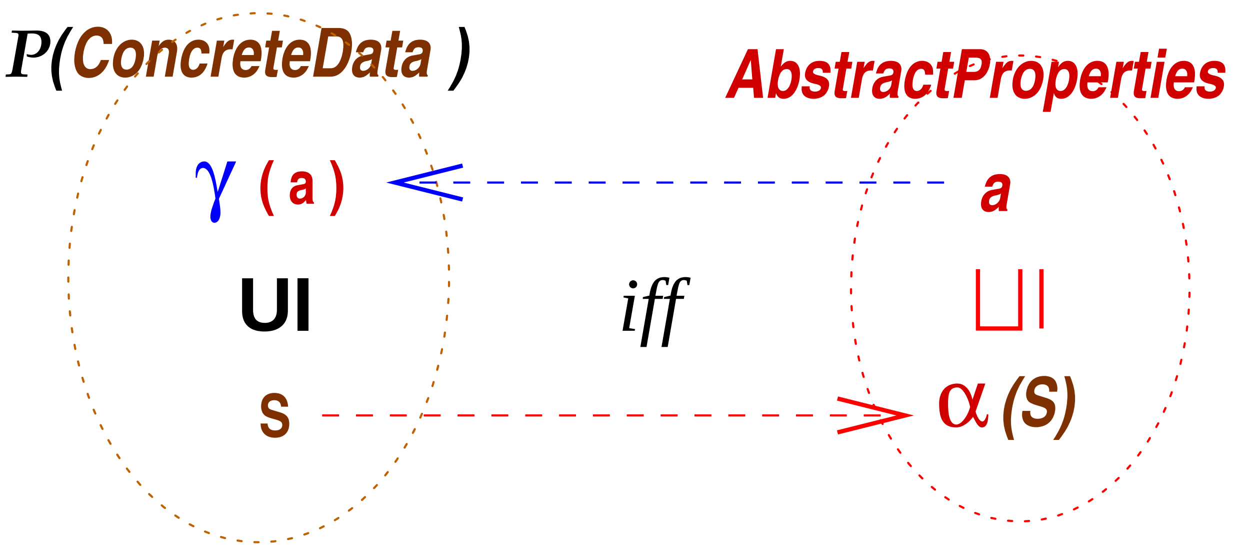
Function $\alpha$ maps each set to the abstract value that best describes it.

## Abstraction followed by concretization demonstrates that $\alpha$ is sound but not exact



Nonetheless, the $\alpha$ given here is as precise as it possibly can be, given the abstract value domain and $\gamma$.

## A *Galois connection* formalizes the situation



That is, for all $S \in \mathcal{P}(\mathrm{ConcreteData}), a \in \mathrm{AbstractProperties}$,

$$S \subseteq \gamma(a) \text{ iff } \alpha(S) \sqsubseteq a$$

When $\alpha$ and $\gamma$ are monotone, this is equivalent to

$$S \subseteq \gamma \circ \alpha(S) \quad \text{and} \quad \alpha \circ \gamma(a) \sqsubseteq a$$

For practical reasons, the second inequality is usually restricted to $\alpha \circ \gamma(a) = a$, meaning that all abstract properties are “exact.”

## Perhaps the oldest application of abstract interpretation is to data-type checking

```
int x;
int[] a = new int[10];
 ...
a[0] = x + 2;  // Whatever x's run-time value might
 ...           //   be, we know it is an  int.
a[1] = (!x);   // Erroneous --- an  int  cannot be
               //   negated, nor can a  bool  be
               //   saved in an  int  cell.
```

## But compilers employ imprecise abstractions

```
int x;
int[] a = new int[10];
 ...              // Because  x's  value is described
a[2 * x] = 3;    //  imprecisely, we cannot decide
                 //  whether  2 * x  falls in the
                 //  interval,  [0,9].
```

We might address array-indexing calculation by

1. making the abstraction more precise, e.g., declaring `x` with the abstract value ("data type") $[0, 9]$;
2. computing a "symbolic execution" of the program with the abstract values

These extensions underlie data-¤ow analyses and many sophisticated program analysis techniques.

## A starting point: Trace-based operational semantics

```
p0 : while isEven(x) {
       p1 : x = x div 2;
     }
p2 : x = 4 * x;
p3 : exit
```

The operational semantics updates a program-point, storage-cell pair, $pp, x$, using these four transition rules:

$$p_0, 2n \longrightarrow p_1, 2n \qquad p_1, n \longrightarrow p_0, n/2$$

$$p_0, 2n+1 \longrightarrow p_2, 2n+1 \qquad p_2, n \longrightarrow p_3, 4n$$

A program's operational semantics is written as a trace:

$$p_0, 12 \longrightarrow p_1, 12 \longrightarrow p_0, 6 \longrightarrow p_1, 6 \longrightarrow p_0, 3 \longrightarrow p_2, 3 \longrightarrow p_3, 12$$

## We can abstractly interpret, say, for parity

```
p0 : while isEven(x) {
       p1 : x = x div 2;
     }
p2 : x = 4 * x;
p3 : exit
```

$$p_0, \mathit{even} \longrightarrow p_1, \mathit{even}$$

$$p_0, \mathit{odd} \longrightarrow p_2, \mathit{odd}$$

$$p_1, \mathit{even} \longrightarrow p_0, \mathit{even}$$

$$p_1, \mathit{even} \longrightarrow p_0, \mathit{odd}$$

$$p_2, a \longrightarrow p_3, \mathit{even}$$

Two trace trees cover the full range of inputs:

$$\begin{tikzcd}
{} \arrow[d] & & {} \arrow[d] \\
\begin{matrix} p_0, \mathit{even} \\ p_1, \mathit{even} \end{matrix} \arrow[loop left] \arrow[d] & & \begin{matrix} p_0, \mathit{odd} \\ p_2, \mathit{odd} \\ p_3, \mathit{even} \end{matrix} \\
\begin{matrix} p_0, \mathit{odd} \\ p_2, \mathit{odd} \\ p_3, \mathit{even} \end{matrix} & & .
\end{tikzcd}$$

The interpretation of the program's semantics with the abstract values is an *abstract interpretation*:

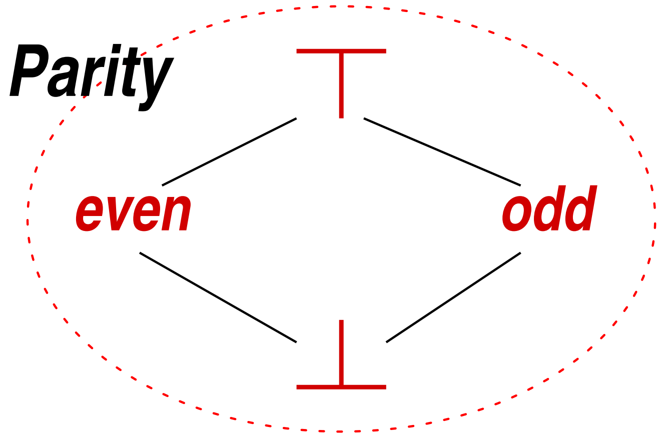
$$\begin{tikzcd}
{} \arrow[d] & & {} \arrow[d] \\
\begin{array}{l} p_0, \mathit{even} \\ p_1, \mathit{even} \end{array} \arrow[loop left] \arrow[dr] & & \begin{array}{l} p_0, \mathit{odd} \\ p_2, \mathit{odd} \\ p_3, \mathit{even} \end{array} \\
& \begin{array}{l} p_0, \mathit{odd} \\ p_2, \mathit{odd} \\ p_3, \mathit{even} \end{array} & .
\end{tikzcd}$$

We conclude that

- if the program terminates, `x` is even-valued
- if the input is odd-valued, the loop body, $p_1$, will not be entered

Due to the loss of precision, we can not decide termination for almost all the even-valued inputs. (Indeed, only $0$ causes nontermination.)

## The underlying abstract-interpretation semantics



$\gamma : \mathrm{Parity} \rightarrow \mathcal{P}(\mathrm{Int})$

$\gamma(\mathrm{even}) = \{..., -2, 0, 2, ...\}$

$\gamma(\mathrm{odd}) = \{..., -1, 1, 3, ...\}$

$\gamma(\top) = \mathrm{Int}, \quad \gamma(\bot) = \{\,\}$

$\alpha : \mathcal{P}(\mathrm{Int}) \rightarrow \mathrm{Parity}$

$\alpha(S) = \sqcup\{\beta(v) | v \in S\}$, where $\beta(2n) = even$ and $\beta(2n+1) = odd$

The abstract transition rules are synthesized from the orginals:

$$p_i, a \longrightarrow p_j, \alpha(v'), \text{ if } v \in \gamma(a) \text{ and } p_i, v \longrightarrow p_j, v'$$

This recipe ensures that every transition in the original, “concrete” semantics is simulated by one in the abstract semantics.

To elaborate, remember that an abstract state, $p_i, a$, represents (abstracts) the set of concrete states,

$$\gamma_{State}(p_i, a) = \{p_i, c \mid c \in \gamma(a)\}$$

So, if some $p_i, c$ in the above set can transit to $p_j, c'$, then its abstraction must make a similar move:

$p_i, c \longrightarrow p_j, c'$ implies $p_i, a \longrightarrow p_j, a'$, where $p_j, c' \in \gamma_{State}(p_j, a')$.

Thus, the abstract semantics simulates all computation traces of the concrete semantics (and due to imprecision, produces more traces than are concretely possible).

Given a Galois connection, $\alpha, \gamma$, we synthesize the most precise abstract semantics that simulates the concrete one as de£ned on the previous slide.

## Abstract interpretation underlies most *static analyses*

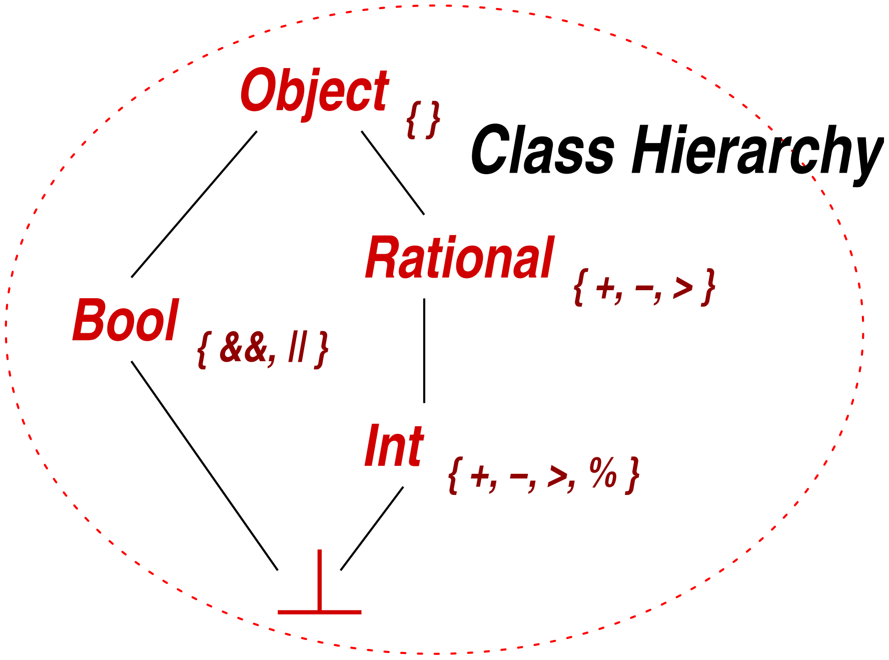
A *static analysis* of a program is a *sound, £nite, and approximate* calculation of the program's executions. The trace trees we just generated for the loop program is an example of a static analysis.

We will survey static analyses for

- data-type inference
- code improvement
- debugging
- assertion synthesis and program proving
- model-checking temporal logic formulas

## Data-type compatibility inference

```
p0 : x = 4;
p1 : while ... {
       p2 : x = (x > 0)
     }
p3 : x = x % 2;
p4 : exit
```



$p_0, \tau \longrightarrow p_1, \mathrm{Int}$

$p_1, \tau \longrightarrow p_2, \tau$

$p_1, \tau \longrightarrow p_3, \tau$

$p_2, \tau \longrightarrow p_1, \mathrm{Bool}, \text{ if } \tau \sqsubseteq \mathrm{Rational}$

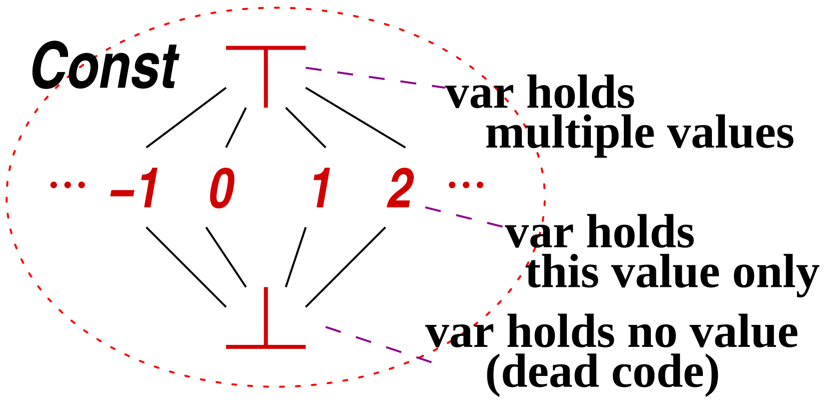
$p_3, \mathrm{Int} \longrightarrow p_4, \mathrm{Int}$

Abstract trace:

$$\begin{tikzcd}
 & & \begin{array}{c} p_0, \mathrm{Object} \\ p_1, \mathrm{Int} \end{array} \arrow[dl] \arrow[dr] & \\
 & \begin{array}{c} p_2, \mathrm{Int} \\ p_1, \mathrm{Bool} \end{array} \arrow[dl] \arrow[dr] & & \begin{array}{c} p_3, \mathrm{Int} \\ p_4, \mathrm{Int} \end{array} \\
\begin{array}{c} p_2, \mathrm{Bool} \\ \mathrm{error} \end{array} & & \begin{array}{c} p_3, \mathrm{Bool} \\ \mathrm{error} \end{array} &
\end{tikzcd}$$

## Constant propagation analysis

```
p0 :  x = 1; y = 2;
p1 :  while (x < y + z)
        p2 : x = x + 1;
      }
p3 :  exit
```


Const
⊤
... -1 0 1 2 ...
⊥
var holds multiple values
var holds this value only
var holds no value (dead code)

where $m + n$ is interpreted

$k_1 + k_2 \longrightarrow sum(k_1, k_2),$

$\top \neq k_i \neq \bot, i \in 1..2$

$\top + k \longrightarrow \top$

$k + \top \longrightarrow \top$

Let $\langle u, v, w \rangle$ abbreviate

$\langle x : u, y : v, z : w \rangle$

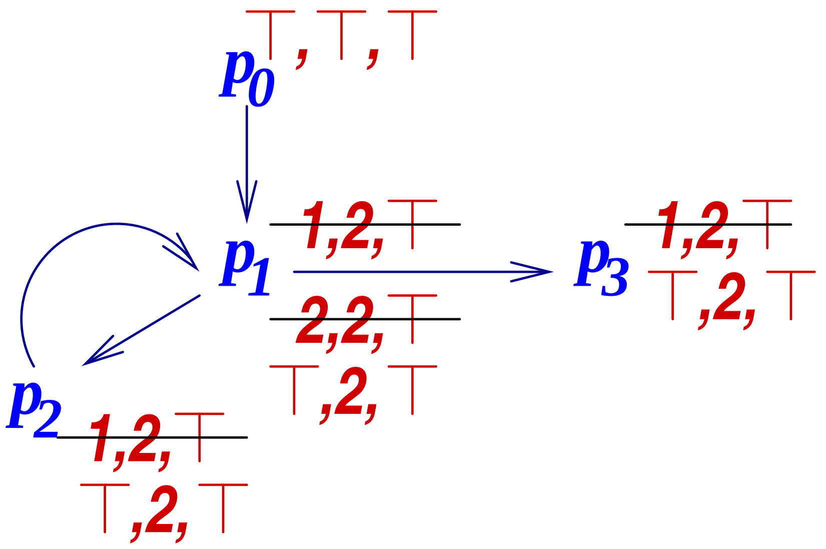
**Abstract trace:**

$$\begin{tikzcd}
p_0, \langle \top, \top, \top \rangle & \\
p_1, \langle 1, 2, \top \rangle \arrow[d] \arrow[dr] & \\
& p_3, \langle 1, 2, \top \rangle \\
p_2, \langle 1, 2, \top \rangle & \\
p_1, \langle 2, 2, \top \rangle \arrow[d] \arrow[dr] & \\
& p_3, \langle 2, 2, \top \rangle \\
p_2, \langle 2, 2, \top \rangle & \\
p_1, \langle 3, 2, \top \rangle \arrow[d] \arrow[dr] & \\
\dots & \dots
\end{tikzcd}$$

## An *acceleration* is needed for £nite convergence

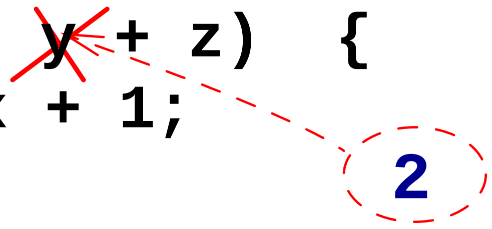
$$\begin{tikzcd}
p_0, \langle \top, \top, \top \rangle & \\
p_1, \langle 1, 2, \top \rangle \arrow[d] \arrow[r] & p_3, \langle 1, 2, \top \rangle \\
p_2, \langle 1, 2, \top \rangle & \\
p_1, \langle 2, 2, \top \rangle \ \sqcup \ \langle 1, 2, \top \rangle & \\
= p_1, \langle \top, 2, \top \rangle \arrow[d] \arrow[r] & p_3, \langle \top, 2, \top \rangle \\
p_2, \langle \top, 2, \top \rangle \arrow[u, bend left] &
\end{tikzcd}$$

**Drawn as a data−flow analysis:**



The analysis tells us to replace $y$ at $p_1$ by 2:

```
p0 :  x = 1; y = 2;
p1 :  while (x < y + z)  {
         p2 :  x = x + 1;
      }
p3 :  exit
```



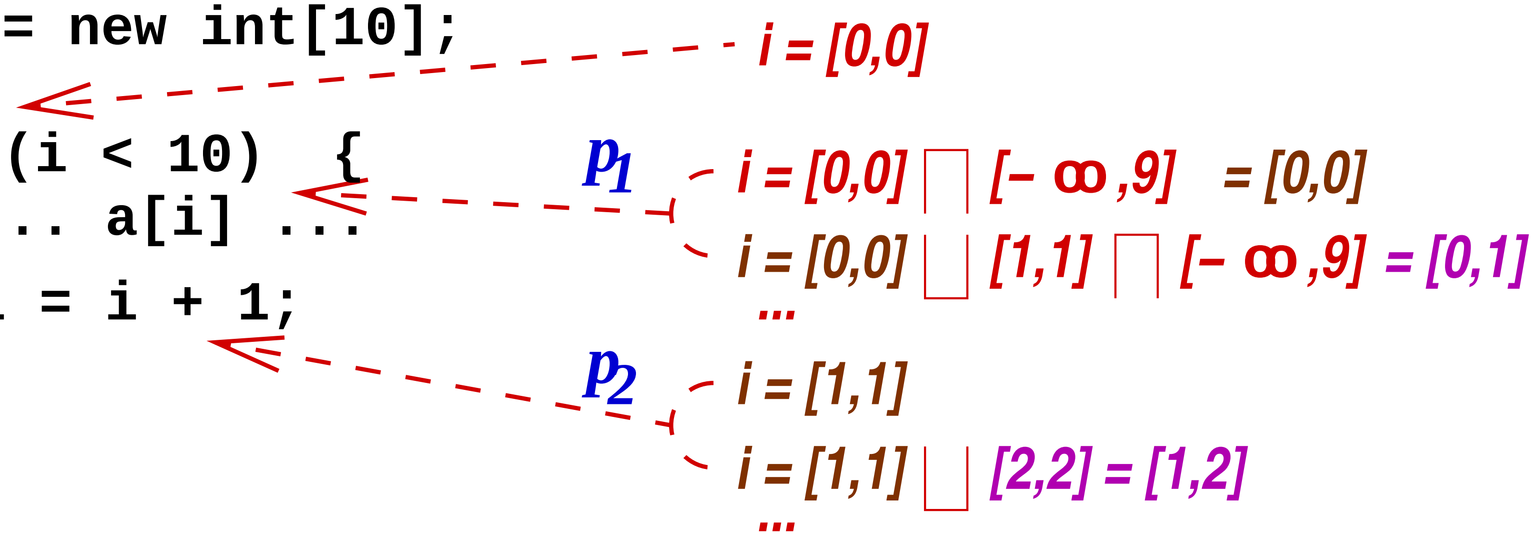
## Array bounds (pre)checking uses intervals

Integer variables receive values from the *interval domain*,

$$I = \{[i,j] \mid i,j \in \mathrm{Int} \cup \{-\infty,+\infty\}\}.$$

We de£ne $[a,b] \sqcup [a',b'] = [\min(a,a'), \max(b,b')]$.

```
int a = new int[10];
i = 0;
while (i < 10)  {
     ... a[i] ...
     i = i + 1;
}
```



$i = [0,0]$
$p_1$
$i = [0,0] \sqcap [-\infty,9] = [0,0]$
$i = [0,0] \sqcup [1,1] \sqcap [-\infty,9] = [0,1]$
...
$p_2$
$i = [1,1]$
$i = [1,1] \sqcup [2,2] = [1,2]$
...

At convergence, $i$'s ranges are

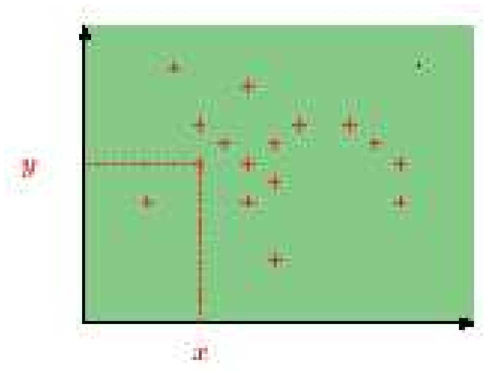
at $p_1$ : $[0..9]$

at $p_2$ : $[1..10]$

at loop exit : $[1..10] \sqcap [10,+\infty] = [10,10]$

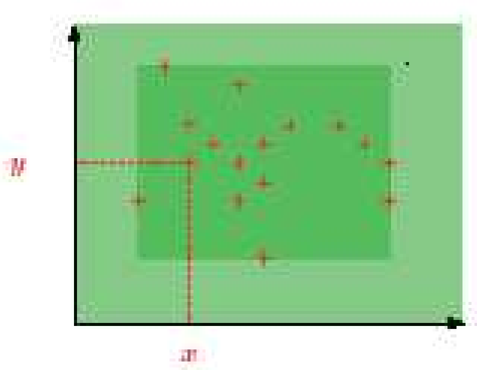
## Examples of relations between variables' values

These Figures are from *Abstract Interpretation: Achievements and Perspectives* by Patrick Cousot, Proc. SSGRR 2000.



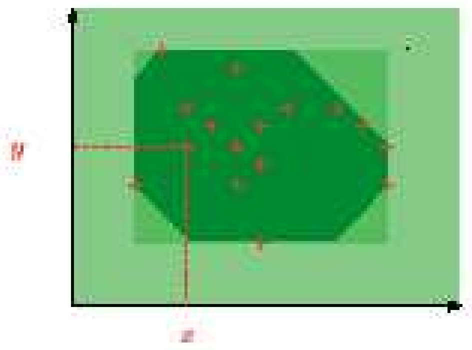
$$\begin{cases} x \geq 0 \\ y \geq 0 \end{cases}$$

Fig. 2
SIGNS



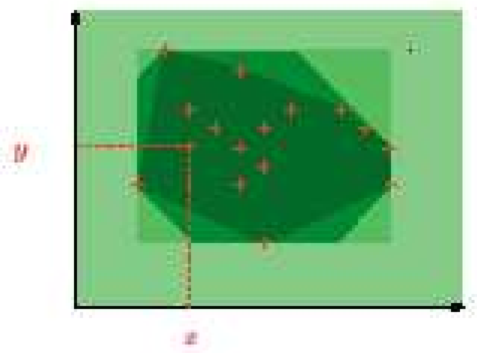
$$\begin{cases} x \in [3,\ 27] \\ y \in [4,\ 32] \end{cases}$$

Fig. 3
INTERVALS



$$\begin{cases} 3 \leq x \leq 27 \\ x + y \leq 88 \\ 4 \leq y \leq 32 \\ x - y \leq 61 \end{cases}$$

Fig. 4
OCTAGONS



$$\begin{cases} 7x + 31y \leq 325 \\ 21x + 7y \geq 0 \end{cases}$$

Fig. 5
POLYHEDRA

## Program veri£cation via *predicate abstraction*

We wish to prove that $z \geq x \wedge z \geq y$ at $p_3$:

```
p0 : if x < y
p1 :   then z = y
p2 :   else z = x
p3 : exit
```

$$\begin{tikzcd}
 & p_0, \langle ?, ?, ? \rangle \arrow[dl] \arrow[dr] & \\
p_1, \langle t, ?, ? \rangle \arrow[d] & & p_2, \langle f, ?, ? \rangle \arrow[d] \\
p_3, \langle t, t, t \rangle & & p_3, \langle f, t, t \rangle
\end{tikzcd}$$

We choose three predicates,

$$\phi_1 = x < y$$

$$\phi_2 = z \geq x$$

$$\phi_2 = z \geq y$$

and compute their values at the program's points. The predicates' values come from the domain, $\{t, f, ?\}$. (Read $?$ as $t \vee f$.)

At all occurrences of $p_3$ in the abstract trace, $\phi_2 \wedge \phi_3$ holds.

## When a goal is undecided, *re£nement* is necessary

Prove $\phi_0 \equiv x \geq y$ at $p_4$:

```
p0 : if !(x >= y)
p1 : then { i = x;
      p2 :  x = y;
      p3 :  y = i;
p4 : }
```

$$\begin{tikzcd}
 & p_0, \langle ? \rangle \arrow[dl] \arrow[dr] & \\
p_1, \langle f \rangle & & p_4, \langle t \rangle \\
p_2, \langle f \rangle & & \\
p_3, \langle t \rangle & & \\
p_4, \langle ? \rangle & &
\end{tikzcd}$$

To decide the goal, we must re£ne the state by adding a needed auxiliary predicate: $wp(y = i, x \geq y) = (x \geq i) \equiv \phi_1$.

$$\begin{tikzcd}
 & p_0, \langle ?, ? \rangle \arrow[dl] \arrow[dr] & & \\
p_1, \langle f, ? \rangle & & p_4, \langle t \rangle & \\
p_2, \langle f, t \rangle & & & \\
p_3, \langle t, t \rangle \arrow[rrr, dash, dashed] & & & \text{because } x \not\geq y \text{ and } x \geq i \text{ imply } y > i \text{ implies } x_{new} \geq i \\
p_4, \langle t, t \rangle \arrow[rrr, dash, dashed] & & & \text{because } x \geq i \text{ implies } x_{new} \geq y
\end{tikzcd}$$

But incremental predicate re£nement cannot synthesize many interesting loop invariants. For this example:

```
p0 : i = n; x = 0;
p1 : while  i != 0  {
   p2 : x = x + 1;  i = i - 1;
     }
p3 : goal: x = n
```

We £nd that the initial predicate set, $P_0 \equiv \{i = 0, x = n\}$, does not validate the loop body.

The £rst re£nement suggests we add $P_1 \equiv \{i = 1, x = n - 1\}$ to the program state, but this fails to validate a loop that iterates more than once.

Re£nement stage $j$ adds predicates $P_j \equiv \{i = j, x = n - j\}$; the re£nement process continues forever!

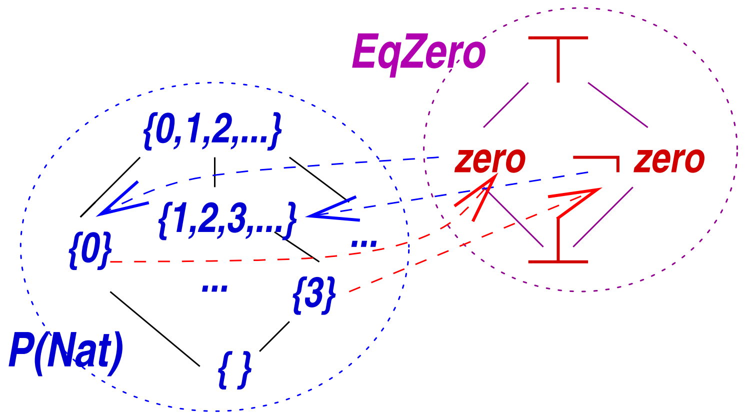
*The loop invariant is* $x = n - i$ :-)

## An abstract domain de£nes a “logic”

For abstract domain $A$, $a \in A$ is a “property/predicate,” and $\gamma(a) \subseteq C$ de£nes a subset of concrete states that make $a$ “true.” For $s \in C$,

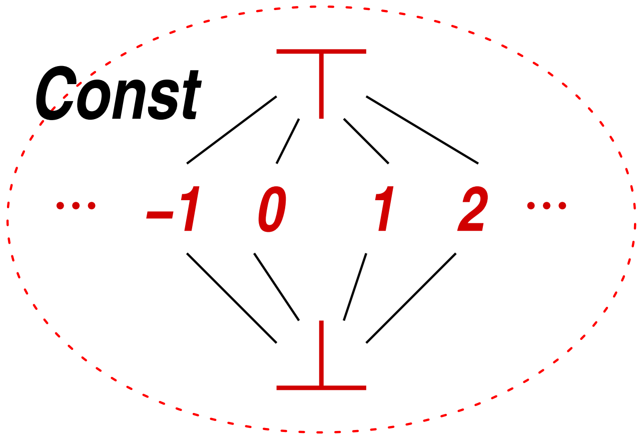
$$s \; has \; a \text{ , written } s \models_A a\text{, iff } s \in \gamma(a) \text{ iff } \alpha\{s\} \sqsubseteq a$$

Example: We might abstract $\mathrm{Nat}$ by $\mathrm{EqZero}$:



We have, for example, that $3 \models \neg\mathrm{zero}$; we also have that $3 \models \top$; and we have that $3 \models \neg\mathrm{zero} \sqcap \top$.

In one sense, *every* analysis based on abstract interpretation is a "predicate abstraction." But the "logic" is weak — it supports conjunction ($\sqcap$) but not necessarily disjunction ($\sqcup$).



For **Const**, we have that

$2 \models_{\mathrm{Const}} 2 \sqcap \top$ ***iff*** $2 \models_{\mathrm{Const}} 2$ *and* $2 \models_{\mathrm{Const}} \top$.

In general, $n \models_{\mathrm{Const}} a \sqcap a'$ iff $n \models_{\mathrm{Const}} a$ and $n \models_{\mathrm{Const}} a'$

But **Const** does not support disjunction: $2 \models_{\mathrm{Const}} \top$, and $\top = 2 \sqcup 3 = 3 \sqcup 4 = 2 \sqcup 3 \sqcup 4$, etc.

Hence $2 \models_{\mathrm{Const}} 3 \sqcup 4$, but this does *not* imply that $2 \models_{\mathrm{Const}} 3$ or $2 \models_{\mathrm{Const}} 4$ !

# Abstract traces can be *model checked*

$p_0$ : `while x > 0 {`

  $p_1$ : *use resource*

   `x = x + 1;  }`

$p_2$ : *sleep forever*

||

$q_0$ : `x = 0;`

$q_1$ : *use resource forever*

$$\begin{tikzcd}
& p_0, q_0, \neg zero \arrow[dl] \arrow[dr] & \\
p_1, q_0, \neg zero \arrow[ur, bend left] \arrow[d] & & p_0, q_1, zero \\
p_1, q_1, zero \arrow[d] & & p_2, q_1, zero \arrow[loop right] \\
p_0, q_1, \neg zero \arrow[d] & & \\
p_1, q_1, \neg zero \arrow[u, bend left] & &
\end{tikzcd}$$

Starting from $p_0, q_0, k$, for $k > 0$, will every execution "Generally/Globally" avoid resource misuse ?

$$p_0, q_0, k \models G\ \neg(p_1 \wedge q_1)\ ?$$

Will every execution reach a Future state where $x$ is permanently (Generally/Globally) zero?

$$p_0, q_0, k \models FG\ zero\ ?$$

The logical operators, F and G, describe reachability properties in the temporal logic, *LTL*.

A state, $s_0$, names the set of traces that begin with it. An LTL property, $\phi$, describes a pattern of states in a trace.

$s_0 \models \phi$ means that all traces, $s_0 \rightarrow s_1 \rightarrow \cdots$, contain pattern $\phi$.

MiniLTL: $\phi ::= \; a \mid G\phi \mid F\phi$ Semantics: $[\![\phi]\!] \subseteq \mathcal{P}(\mathit{Trace})$

$$[\![a]\!] \;=\; \{\pi \mid \pi_0 \models_A a\}$$

$$[\![G\phi]\!] \;=\; \{\pi \mid \forall i \geq 0, \pi \downarrow i \in [\![\phi]\!]\}$$

$$[\![F\phi]\!] \;=\; \{\pi \mid \exists i \geq 0, \pi \downarrow i \in [\![\phi]\!]\}$$

where, for $\pi = s_0 \rightarrow s_1 \rightarrow \cdots$, let $\pi_0 = s_0$ and $\pi \downarrow i = s_i \rightarrow s_{i+1} \rightarrow \cdots$.

There is a Galois connection, $(\mathcal{P}(\mathit{Trace}), \subseteq) \leftrightarrow (\mathcal{P}(\text{MiniLTL}), \supseteq)$, where $\sqcup = \cap$ in $\mathcal{P}(\text{MiniLTL})$:

$\gamma(P) = \bigcap\{[\![\phi]\!] \mid \phi \in P\}$ – the traces that have all the properties in $P$

$\alpha(S) = \{\phi \mid S \subseteq [\![\phi]\!]\}$ – properties held by all traces in $S$

But this is just the beginning of a long story about the relationship of abstract interpretation to temporal-logic model checking!

## Every concrete value is the conjunction of its abstractions *(its "abstract-interpretation DNA")*

$$= \mathrm{elephant}_{\mathrm{species}} \wedge \mathrm{brown}_{\mathrm{color}} \wedge \mathrm{heavy}_{\mathrm{weight}} \wedge 4000..6000\mathrm{kg}_{\mathrm{weight}} \wedge \cdots$$

There is even a pattern of Galois connection for this:

$$\gamma : \mathrm{AllPossibleProperties} \to \mathcal{P}(\mathrm{RealWorldObjects})$$
$$\gamma(p) = \{c \in \mathrm{RealWorldObjects} \mid c \textit{ has property } p\}$$

$$\beta : \mathrm{RealWorldObjects} \to \mathrm{AllPossibleProperties}$$
$$\beta(c) = \sqcap\{p \in \mathrm{AllPossibleProperties} \mid c \in \gamma(p)\}$$

$$\alpha : \mathcal{P}(\mathrm{RealWorldObjects}) \to \mathrm{AllPossibleProperties}$$
$$\alpha(S) = \sqcup\{\beta(s) \mid s \in S\}$$

## References

- The papers of Patrick and Radhia Cousot (`www.di.ens.fr/~cousot`), including
  1. Abstract interpretation: a uni£ed lattice model for static analysis of programs by construction or approximation of £xpoints. ACM POPL 1977.
  2. Systematic design of program analysis frameworks. ACM POPL, 1979.
  3. Abstract interpretation: achievements and perspectives. Proc. SSGRR 2000.
- Neil Jones and Flemming Nielson. Abstract Interpretation: a Semantics-Based Tool for Program Analysis. In Handbook of Logic in Computer Science, Vol. 4, Oxford University Press, 1994.
- Hanne Nielson, Flemming Nielson, and Chris Hankin. Principles of Program Analysis. Springer 1999.
- A few of my papers, found at `www.cis.ksu.edu/~schmidt/papers`:
  1. Trace-Based Abstract Interpretation of Operational Semantics. J. Lisp and Symbolic Computation 10-3 (1998).
  2. Data-¤ow analysis is model checking of abstract interpretations. ACM POPL 1998.