Abstract Interpretation from a Topological Perspective

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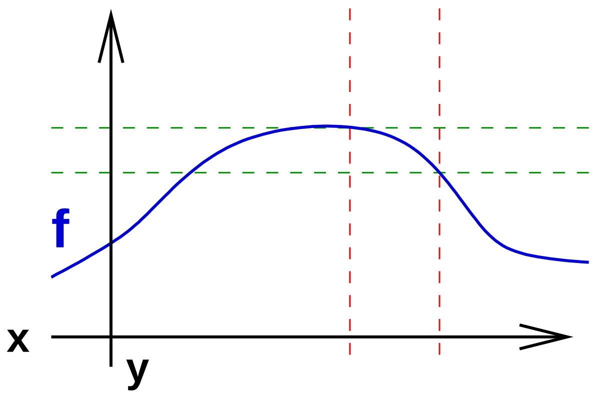
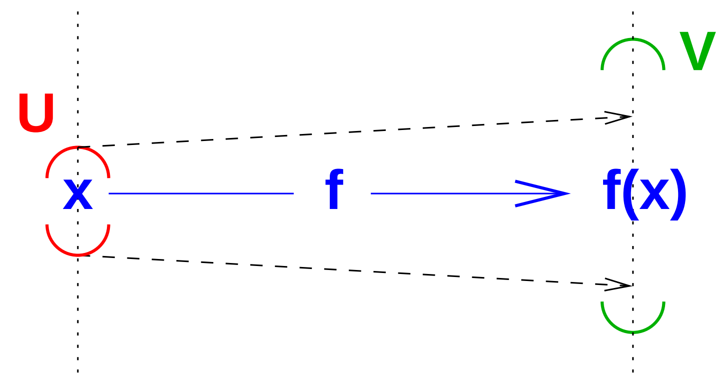
Motivation and overview of results

Topology *studies convergent mappings*



As argument x converges within smaller and smaller open invervals, so does answer $f(x)$.

In topology, "open interval" (a property of interest) generalizes to *open set*; "convergent function" generalizes to *continuous function*:



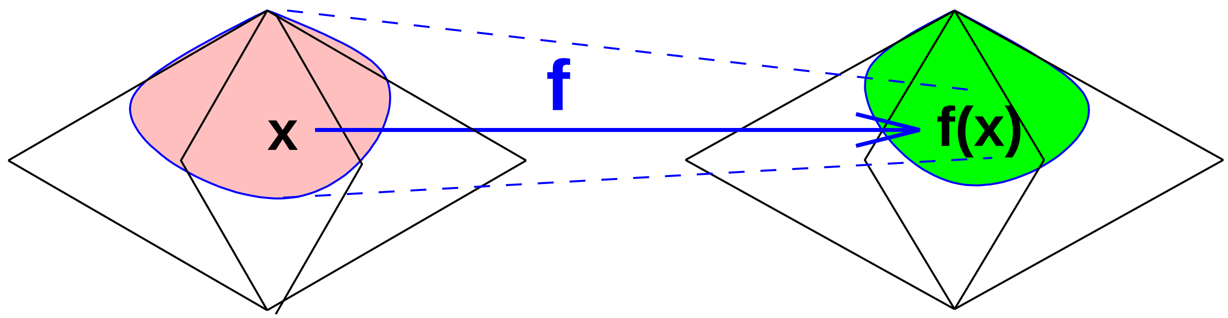
Let $\mathcal{O}_\Sigma$ be the open sets. $f : \Sigma \to \Sigma$ is *(topologically) continuous* iff for all $x \in \Sigma$ and $V \in \mathcal{O}_\Sigma$, if $f(x) \in V$, then there exists some $U \in \mathcal{O}_\Sigma$ such that $x \in U$ and $f[U] \subseteq V$.

Domain theory uses the Scott topology

For an algebraic lattice, $(\Sigma, \sqsubseteq)$, the Scott-open sets are those $U \subseteq \Sigma$ such that U is

- *upwards closed:* if $c \in U, c \sqsubseteq d$, then $d \in U$ also.
- *closed under tails of chains:* for every chain $C \subseteq \Sigma$, if $\sqcup C \in U$, then $\exists c \in C$ such that $c \in U$ also.

(A Scott-open U is like an interval, $(c, +\infty]$, on the real line.)



$f : \Sigma \longrightarrow \Delta$ is Scott-topologically continuous iff it is chain continuous (i.e., for every chain $C \subseteq \Sigma$, $f(\sqcup C) = \sqcup\{f(c) \mid c \in C\}$).

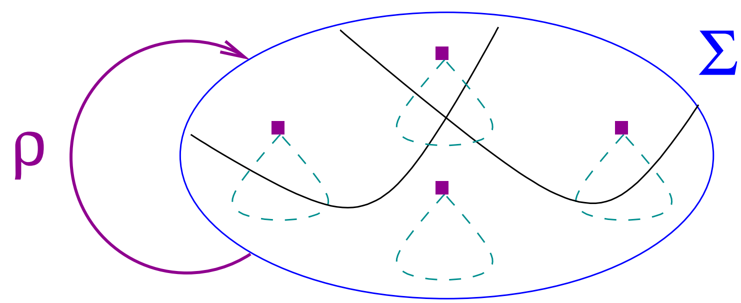
The Scott topology on algebraic lattice D *defines* D itself.

An application to abstract interpretation: Cousots' $\sqcup$-topology for a.i.

[Cousot²78] defines a Scott-like topology for complete lattices, where the basic open sets are up-closed, closed under tails of chains, and closed under binary $\sqcap$. They show equivalence of chain continuity to topological continuity.

Next, they show that an abstract interpretation on Σ, defined by an upper-closure map, $\rho : \Sigma \rightarrow \Sigma$, preserves convergence. That result follows from this key property:

Proposition: The $\sqcup$-topology on $\rho[\Sigma]$ is exactly the *relative $\sqcup$-topology* on Σ, that is, every $V \in \mathcal{O}_{\rho[\Sigma]}$ equals $U \cap \rho[\Sigma]$, for some $U \in \mathcal{O}_\Sigma$.

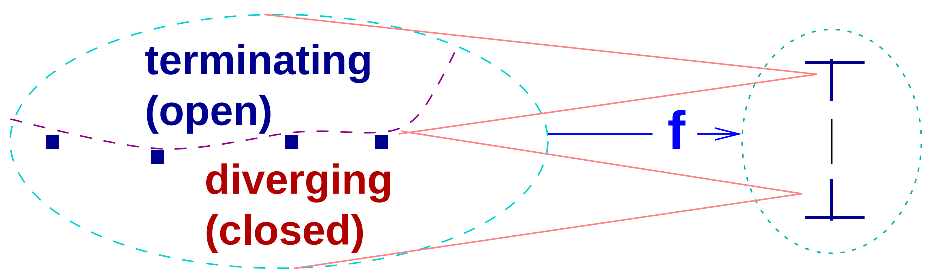


An application: Backwards strictness analysis

$f : D_\perp \to D_\perp$ is *strict*, if $f(\perp) = \perp$. This knowledge can help optimize code for f for lazy functional languages [Mycroft80] .

A backwards strictness analysis approximates $D_\perp$ by $2_D = \{\top, \perp\}$ and and f by $f^\sharp : D_\perp \to 2_D$, computing ${f^\sharp}^{-1}\{\top\}$ for termination information.

[Clack&PeytonJones85] showed how to use a finite set of minimal points (a *frontier*) to represent ${f^\sharp}^{-1}\{\top\}$.



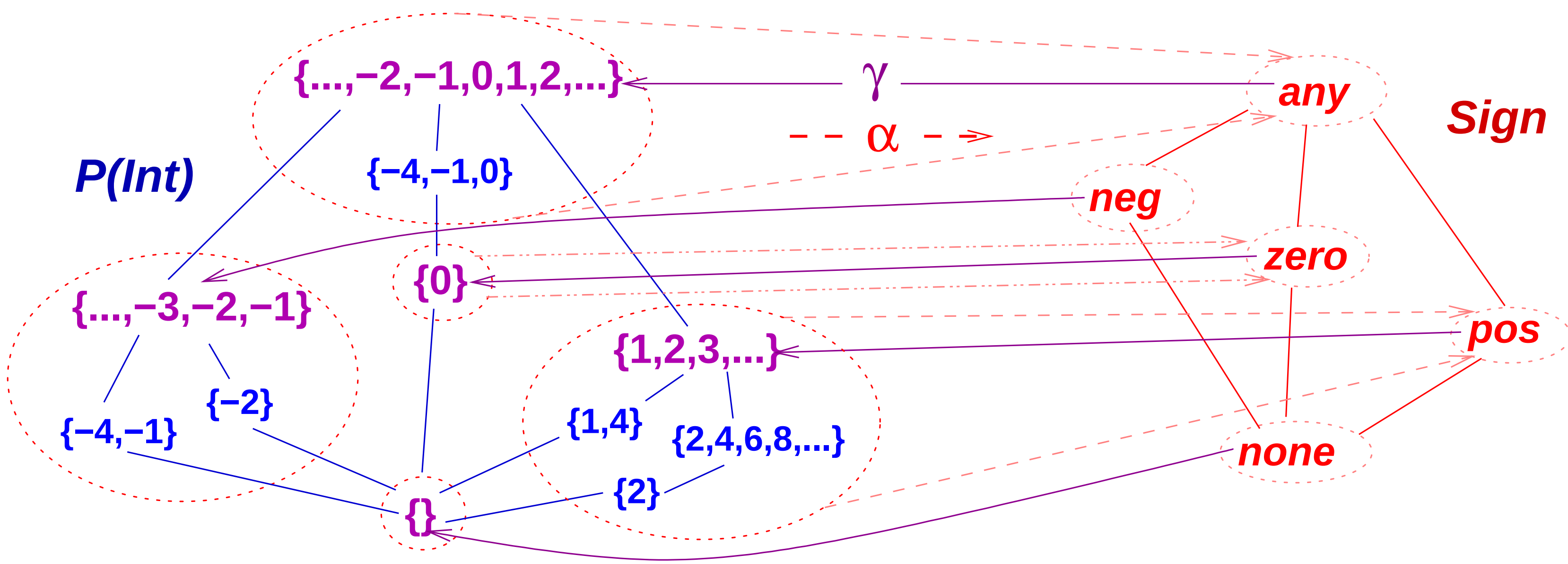
[Hunt89] noted that $f^\sharp$ is Scott-continuous, making ${f^\sharp}^{-1}\{\top\}$ a Scott-open set. Hunt defined frontier-based strictness analysis as calculation of open-set inverse images, showing how to lift efficiently to higher types, in the style of [BurnHankinAbramsky86] .

An application: abstract interpretation in logical form

[Abramsky91] applied frame theory (the axiomatization of the lattice of open sets) to domain theory, generating Scott domains from sets of atomic elements that act as primitive propositions in a domain logic, closing them under a set of frame axioms.

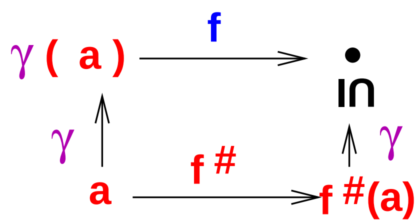
[Jensen92] observed one can use a *finite subset* of a domain's atomic elements to generate an *abstract domain* that approximates the concrete domain generated from *all* the atomic elements. Jensen called his methodology *abstract interpretation in logical form* and applied it to strictness analysis.

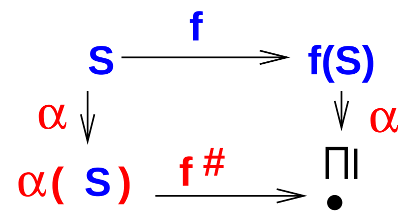
We step back from these applications and ask: *In what naive sense does an abstract domain define a "topology" on the concrete domain that it approximates?* What does it mean for a function to preserve and reflect the "open sets"? Do these notions define forwards and backwards static analyses and do they ensure soundness and completeness of the analyses?



Here, does $\gamma[Sign] = \{\{\}, \{\cdots, -2, -1\}, \{0\}, \{1, 2, \cdots\}, Int\}$ *define a "topology" on* Int*?*

For Galois connection, $\mathcal{P}(\Sigma)\langle\alpha,\gamma\rangle A$, for $f : \mathcal{P}(\Sigma) \rightarrow \mathcal{P}(\Sigma)$, an abstract function, $f^\sharp : A \rightarrow A$, is ***sound*** iff $f \circ \gamma \sqsubseteq \gamma \circ f^\sharp$ iff $\alpha \circ f \sqsubseteq f^\sharp \circ \alpha$:



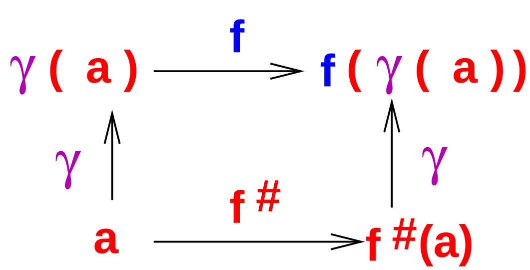


α and γ are semi-homomorphisms; $f^\sharp$ is a postcondition transformer on A; $f_0^\sharp = \alpha \circ f \circ \gamma$ is strongest.

Using $\rho = \gamma \circ \alpha$, we can define $f_0^\sharp = \rho \circ f : \gamma[\Sigma] \rightarrow \gamma[\Sigma]$.

Forwards completeness

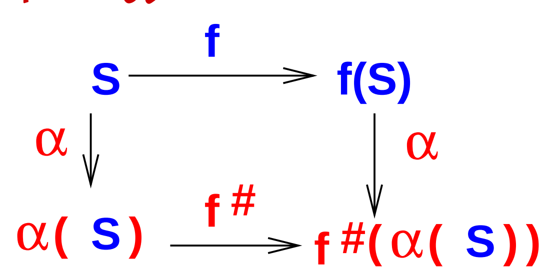
[Giacobazzi01] : $f \circ \gamma = \gamma \circ f^\sharp$



Backwards completeness

[Cousot²79,Giacobazzi00] :

$\alpha \circ f = f^\sharp \circ \alpha$



$f_0^\sharp$ is forwards complete for f iff $f \circ \rho = \rho \circ f \circ \rho$.

$f_0^\sharp$ is backwards complete for f iff $\rho \circ f = \rho \circ f \circ \rho$

A key characterization from *[GiacobazziQuintarelli01]*

The following result is the main result in [14] and it is the basis for a constructive characterization of the complete shell of an abstract domain, viz. the least (most abstract) domain which is $\mathcal{B}$-complete and includes a given domain. This result constructively characterizes the structure of $\mathcal{B}$-complete abstract domains for continuous functions. Recall that, if $f : C \to C$ is a unary function, then $f^{-1}(y) = \{\, x \mid f(x) = y \,\}$.

Theorem 1 ([14]). *Let $f : C \to C$ be continuous and $\rho \in uco(C)$. Then ρ is $\mathcal{B}$-complete for f iff $\bigcup_{y \in \rho(C)} max(f^{-1}(\downarrow y)) \subseteq \rho(C)$.*

This characterization of backwards completeness looks like the inverse-image definition of topological continuity, stated in a kind of frame theory.

Is a.i. completeness the same as topological continuity?

Where the current paper fits into this story...

Topology studies how functions compute on *properties* (open sets). This is exactly what abstract interpretation studies.

We proceed from these first principles: For concrete domain, Σ, abstract domain, A, and concretization map, $\gamma : A \rightarrow \mathcal{P}(\Sigma)$,

1. A defines a *property family*, $\mathcal{F}_\Sigma = \gamma[A] \subseteq \mathcal{P}(\Sigma)$.
2. if $\mathcal{F}_\Sigma$ is closed under intersection, it is a *closed family* (call it $\mathcal{C}_\Sigma$); if $\mathcal{F}_\Sigma$ is closed under union, it is an *open family* (call it $\mathcal{O}_\Sigma$).
3. For $f : \Sigma \rightarrow \Sigma$, we generalize the definition of continuity: f is $\mathcal{F}_\Sigma$-*continuous* iff for all $S \subseteq \Sigma$ and $U \in \mathcal{F}_\Sigma$,
   if $f[S] \subseteq U$, then there exists $V \in \mathcal{F}_\Sigma$ such that
   $S \subseteq V$ and $f[V] \subseteq U$.

We gain these results

1. Generalized continuity retains fundamental properties. In particular, f is continuous iff $f^{-1}[U] \in \mathcal{F}_\Sigma$ whenever $U \in \mathcal{F}_\Sigma$.
2. Closed families generate forwards abstract interpretations with best (strongest postcondition) precision (e.g., constant propagation), and open families generate backwards abstract interpretations with best (weakest precondition) precision (e.g., strictness analysis).
3. [Giacobazzi00 and 01] 's notions of forwards and backwards completeness are characterized as the topologically closed maps and topologically continuous maps upon a closed family. (There are analogous results for an open family).
4. [Smyth83] 's upper and lower topologies for powerdomains $\mathcal{P}(\Sigma)$ generate the abstract interpretations based on $\mathcal{F}_\Sigma$ for abstract-model checking of $\Box$ and $\Diamond$ in branching-time temporal logic.

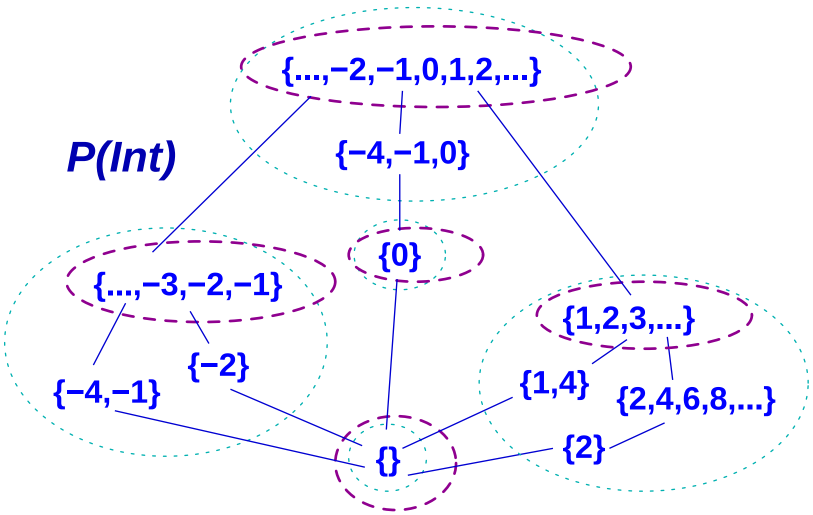
Technical details

Open sets are computable properties [Smyth83]

For an algebraic cpo, D, its Scott-basic-open sets are $\uparrow\mathrm{e}$, for each finite element, $\mathrm{e} \in \mathrm{D}$. *Read* $\mathrm{d} \in \uparrow\mathrm{e}$ *as "*d *has property* $\uparrow\mathrm{e}$*."*

But abstract intepretation is *finite computation on properties*; for an abstract domain, like $Sign$, the computable properties are $\gamma[Sign]$ (or, if you will, $\rho[\mathcal{P}(Sign)]$, where $\rho = \gamma \circ \alpha$).

Alas, $\rho[\mathcal{P}(Sign)]$ is closed under intersections (not necessarily unions). Also, there exist abstract domains A that possess *only* a γ but no α (and no ρ) [Cousot^{2}92] .



Let's weaken some definitions

For abstract domain A and $\gamma : A \rightarrow \mathcal{P}(\Sigma)$, define Σ's *property family* as $\mathcal{F}_\Sigma = \gamma[A]$.

For each $U \in \mathcal{F}_\Sigma$, its complement is $\sim U = \Sigma - U$; for $\mathcal{F}_\Sigma$, its *complement family*, $\sim\mathcal{F}_\Sigma$, is $\{\sim U \mid U \in \mathcal{F}_\Sigma\}$.

$\mathcal{F}_\Sigma$ is an *open family* if it is closed under unions; it has an interior operation, $\iota : \mathcal{P}(\Sigma) \rightarrow \mathcal{F}_\Sigma$. It is a *closed family* if it is closed under intersections; it has a closure operation, $\rho : \mathcal{P}(\Sigma) \rightarrow \mathcal{F}_\Sigma$. If $\mathcal{F}_\Sigma$ is an open family, then its complement is a closed family (and vice versa).

When γ is the upper adjoint of a Galois connection, then $\mathcal{F}_\Sigma$ is a closed family.

$f^\sharp : \mathcal{F}_\Sigma \rightarrow \mathcal{F}_\Sigma$ is *(overapproximating) sound for* $f : \Sigma \rightarrow \Sigma$ if for all $U \in \mathcal{F}_\Sigma$, $f[U] \subseteq f^\sharp[U]$.

When $\mathcal{F}_\Sigma$ is a closed family, $\rho \circ f$ is sound for f.

There are the obvious dual notions for *underapproximating soundness*.

If $\mathcal{C}_\Sigma$ is a closed family, its closure operator, ρ, defines a strongest-postcondition analysis:

For $f : \Sigma \to \Sigma$, define $f^\sharp : \mathcal{C}_\Sigma \to \mathcal{C}_\Sigma$ as $f^\sharp = \rho \circ f$. We have

$$\{\phi\}f\{f^\sharp(\phi)\}$$

holds true (where $\{\phi\}f\{\psi\}$ asserts $f[\phi] \subseteq \psi$, for $\phi, \psi \in \mathcal{C}_\Sigma$).

$f^\sharp(\phi) = \rho(f[\phi])$ defines the strongest postcondition of f and ϕ expressible in $\mathcal{C}_\Sigma$.

If we desire preconditions from a closed family, then we must close it under unions, that is, perform a *disjunctive completion* of the abstract domain — *We use the closed family as a base for a topology* on Σ, namely, $\{\cup T \mid T \subseteq \mathcal{C}_\Sigma\}$, which is both an open *and* a closed family.

If we are truly interested in preconditions, we start with an *open* family of properties (one closed under unions), $\mathcal{O}_\Sigma \subseteq \mathcal{P}(\Sigma)$, so we have an interior operation, $\iota : \mathcal{P}(\Sigma) \to \mathcal{O}_\Sigma$.

We underapproximate the inverses of transition functions: For $f : \Sigma \to \Sigma$, define $f^{-o} : \mathcal{O}_\Sigma \to \mathcal{O}_\Sigma$ as $f^{-o} = \iota \circ f^{-1}$. This implies

$$\{f^{-o}(\psi)\}f\{\psi\}$$

holds true and $f^{-o}(\psi)$ is the *weakest precondition of* f *and* ψ *expressible in* $\mathcal{O}_\Sigma$.

Proposition: For closed family $\mathcal{C}_\Sigma$ and $\mathcal{O}_\Sigma = \sim\mathcal{C}_\Sigma$, $\widetilde{(f^{-1})^\sharp}(U) = f^{-o}(U)$, for all $U \in \mathcal{O}_\Sigma$. (Note: $\widetilde{(f^{-1})^\sharp} = \sim \circ (f^{-1})^\sharp \circ \sim$.)

That is, by using $\mathcal{C}_\Sigma$'s closure operator to define the overapproximating $(f^{-1})^\sharp$, we can compute an *underapproximating*, weakest-precondition analysis on $\mathcal{O}_\Sigma = \sim\mathcal{C}_\Sigma$ defined as $\widetilde{(f^{-1})^\sharp}$.

Property preservation by functions

For $f : \Sigma \to \Sigma$, define $f : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ as $f[S] = \{f(s) \mid s \in S\}$, and define $f^{-1} : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ as $f^{-1}(T) = \{s \in \Sigma \mid f(s) \in T\}$, as usual.

f is $\mathcal{F}_\Sigma$-*preserving* iff for all $U \in \mathcal{F}_\Sigma$, $f[U] \in \mathcal{F}_\Sigma$. In such a case, $f : \mathcal{F}_\Sigma \to \mathcal{F}_\Sigma$ is well defined.

This generalizes the notions of topologically open and closed maps.

Let $\mathcal{F}_\Sigma$ be a closed family, and let $\rho : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ be the associated closure operator.

For $f : \Sigma \to \Sigma$, define $f_0^\sharp : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ as $f_0^\sharp = \rho \circ f$, as usual.

Theorem: $f_0^\sharp$ is forwards complete for f iff f is $\mathcal{F}_\Sigma$-preserving, that is, iff f is a topologically closed map.

Property reflection (function continuity)

Let U_c (respectively, U_S) denote a member of $\mathcal{F}_\Sigma$ such that $c \in U_c$ (respectively, $S \subseteq U_S$):

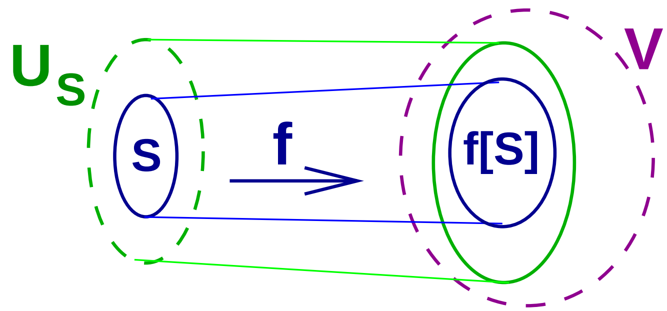
- For $c \in \Sigma$, $f : \Sigma \to \Sigma$ is *continuous at* c iff for all $V_{f(c)} \in \mathcal{F}_\Sigma$, there exists some $U_c \in \mathcal{F}_\Sigma$ such that $f[U_c] \subseteq V_{f(c)}$.
- For $S \subseteq \Sigma$, f is *continuous at* S iff for all $V_{f[S]} \in \mathcal{F}_\Sigma$, there exists some $U_S \in \mathcal{F}_\Sigma$ such that $f[U_S] \subseteq V_{f[S]}$.
- f is $\mathcal{F}_\Sigma$*-reflecting* iff for all $V \in \mathcal{F}_\Sigma$, $f^{-1}(V) \in \mathcal{F}_\Sigma$, that is, f^{-1} is $\mathcal{F}_\Sigma$-preserving.

The second item is needed because $\mathcal{F}_\Sigma$ might not be an open family.

If $\mathcal{F}_\Sigma$ is a topology, then all three notions are equivalent.

reflection, cont.

f is continuous at $S \subseteq \Sigma$:



If $f[S] \subseteq V \in \mathcal{F}_\Sigma$, then there exists $U_S \in \mathcal{F}_\Sigma$ such that $f[U_S] \subseteq V$.

Proposition:

1. f is $\mathcal{F}_\Sigma$-reflecting iff f is continuous at S, for all $S \subseteq \Sigma$.
2. If $\mathcal{F}_\Sigma$ is an open family, then f is $\mathcal{F}_\Sigma$-reflecting iff f is continuous at c, for all $c \in \Sigma$.
3. $f : \Sigma \to \Sigma$ is $\sim\mathcal{F}_\Sigma$-reflecting iff f is $\mathcal{F}_\Sigma$-reflecting.

reflection, concl.

For $S, S' \subseteq \Sigma$, write $S \leq_{\mathcal{F}_\Sigma} S'$ iff for all $K \in \mathcal{F}_\Sigma, S \subseteq K$ implies $S' \subseteq K$. Write $S \equiv_{\mathcal{F}_\Sigma} S'$ iff $S \leq_{\mathcal{F}_\Sigma} S'$ and $S' \leq_{\mathcal{F}_\Sigma} S$. That is, S and S' share the same properties.

Definition: $f : \Sigma \to \Sigma$ is *backwards-$\mathcal{F}_\Sigma$-complete* iff for all $S, S' \subseteq \Sigma$, $S \equiv_{\mathcal{F}_\Sigma} S'$ implies $f[S] \equiv_{\mathcal{F}_C} f[S']$ cf. Slide 12.

Proposition: If f is $\mathcal{F}_\Sigma$-reflecting, then it is backwards-$\mathcal{F}_\Sigma$-complete.

Lemma: If $\mathcal{F}_\Sigma$ is a closed family, then TFAE:
(i) f is backwards-$\mathcal{F}_\Sigma$-complete;
(ii) for all $S \subseteq \Sigma$, $f[S] \equiv_{\mathcal{F}_\Sigma} f[\rho(S)]$;
(iii) $\rho \circ f = \rho \circ f \circ \rho$

Theorem: For closed family, $\mathcal{F}_\Sigma$, f is backwards-$\mathcal{F}_\Sigma$-complete iff it is $\mathcal{F}_\Sigma$-reflecting.

So, abstract-interpretation backwards completeness is topological continuity.

What about open families?

Let $\mathcal{F}_\Sigma$ be open (closed under unions) and $\iota : \mathcal{P}(\Sigma) \to \mathcal{F}_\Sigma$ be its interior map.

We use an open family to perform an underapproximating *precondition analysis*: for $f : \Sigma \to \Sigma$, define $f^{-1} : \mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ as $f^{-1}(S) = \{s \in \Sigma \mid f(s) \in S\}$, as usual.

The strongest (*weakest precondition*) abstract function for f^{-1} is $\iota \circ f^{-1} : \mathcal{F}_\Sigma \to \mathcal{F}_\Sigma$.

Define *forwards-$\mathcal{F}_\Sigma$-completeness:* $f^{-1} \circ \iota = \iota \circ f^{-1} \circ \iota$
backwards-$\mathcal{F}_\Sigma$-completeness: $\iota \circ f^{-1} = \iota \circ f^{-1} \circ \iota$

f^{-1} is $\mathcal{F}_\Sigma$-preserving iff f^{-1} is forwards-$\mathcal{F}_\Sigma$-complete iff f is $\sim\mathcal{F}_\Sigma$-reflecting iff f is $\mathcal{F}_\Sigma$-reflecting.

This is the classic pre- post-condition duality of predicate transformers.

Powerdomains

[Smyth83] showed, for algebraic Scott-domain D, that the lower powerdomain, $\mathcal{P}_L(D)$, and the upper powerdomain, $\mathcal{P}_U(D)$, are generated from D's Scott-topology, $\mathcal{O}_D$, as follows:

- *lower topology,* $\mathcal{O}^L_{\mathcal{O}_D}$: generated from the base $\mathcal{B}^L_{\mathcal{O}_D} = \{\exists U \mid U \in \mathcal{O}_D\}$, where $\exists U = \{S \subseteq D \mid S \cap U \neq \emptyset\}$ ("all sets that meet U")
- *upper topology,* $\mathcal{O}^U_{\mathcal{O}_D}$: generated from the base $\mathcal{B}^U_{\mathcal{O}_D} = \{\forall U \mid U \in \mathcal{O}_D\}$, where $\forall U = \{S \subseteq D \mid S \subseteq U\}$ ("all sets covered by U")

We can show that when Σ is abstractly interpreted by closed family, $\mathcal{C}_\Sigma$, the abstract interpretation for ACTL checking that proves the most $\mathcal{C}_\Sigma$-properties is generated from the (co)base $\mathcal{B}^U_{\mathcal{C}_\Sigma}$.

Similarly, the best abstract interpretation for ECTL checking is generated from the (co)base $\mathcal{B}^L_{\mathcal{C}_\Sigma}$.

For $f : \Sigma \to \mathcal{P}(\Sigma)$, there are two preimage maps:

1. $\widetilde{pre}_f(S) = \{c \in \Sigma \mid f(c) \subseteq S\}$
2. $pre_f(S) = \{c \in \Sigma \mid f(c) \cap S \neq \emptyset\}$

Abstract model checking [CleavelandIyerYankelevich95, DamsGerthGrumberg97] starts from $\mathcal{C}_\Sigma$ to generate an a.i. for $\mathcal{P}(\Sigma)$:

Let $\mathcal{C}^{U}_{\mathcal{C}_\Sigma}$ be the closed family generated from $\mathcal{B}^{U}_{\mathcal{C}_\Sigma}$:

Theorem: For $\mathcal{C}^{U}_{\mathcal{C}_\Sigma}$, $\widetilde{pre}_f$ is a $\mathcal{C}_\Sigma$-preserving map iff f is $\mathcal{C}_\Sigma\mathcal{C}^{U}_{\mathcal{C}_\Sigma}$-reflecting.

That is, $[f]$ can be precisely model checked exactly when f is $\mathcal{C}_\Sigma\mathcal{C}^{U}_{\mathcal{C}_\Sigma}$-continuous.

Let $\mathcal{C}^{L}_{\mathcal{C}_\Sigma}$ be the closed family generated from $\mathcal{B}^{L}_{\mathcal{C}_\Sigma}$:

Theorem: For $\mathcal{C}^{L}_{\mathcal{C}_\Sigma}$, pre_f is a $\mathcal{C}_\Sigma$-preserving map iff f is $\mathcal{C}_\Sigma\mathcal{C}^{L}_{\mathcal{C}_\Sigma}$-reflecting.

That is, $\langle f \rangle$ can be precisely model checked exactly when f is $\mathcal{C}_\Sigma\mathcal{C}^{L}_{\mathcal{C}_\Sigma}$-continuous.

This is the origin of Dams's mixed Kripke structures.

Conclusion

...many key notions and theorems from abstract interpretation theory appear as definitions and corollaries of “naive topology”...

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