Internal and External Logics of Abstract Interpretations

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Motivations

How does a static analysis "connect" to the properties it is meant to prove?

- use data-flow analysis to compute available-expression sets to decide register allocation;
- use a state-space exploration to model check a temporal-logic safety property or program-transformation criterion
- apply predicate abstraction with counter-example-guided refinement (CEGAR) to generate an assertion set that proves a safety property

The value domain used by an analysis and the logic used for validation/transformation should be *one and the same* — the logic is *internal* to the value domain. If the values and logic differ, then the logic must be defined *externally*.

Developments from this paper

Let Σ be the program's state set; let A be the abstract domain; let $\gamma : A \rightarrow \mathcal{P}(\Sigma)$ be the *concretization function*.

1. γ defines a logic *internal* to A for Σ, where A's elements act *both* as computational values and as logical assertions. The model theory, $\models$, is defined by γ; the proof theory, $\vdash$, by $\sqsubseteq_A$.
2. The notion of (forwards) completeness from abstract interpretation theory *characterizes* the internal logic.
3. When a logic for Σ is proposed independently from γ, then an *external logic* must be fashioned from $\mathcal{P}_{\downarrow}(A)$. *But,* when γ preserves meets and joins, the external logic can be embedded within A^{op} (inverted).

In the last case, A has *two* interpretations: an overapproximating, *computational* interpretation, and an underapproximating *logical* interpretation (on A^{op}).

Abstract interpretation: computing on properties

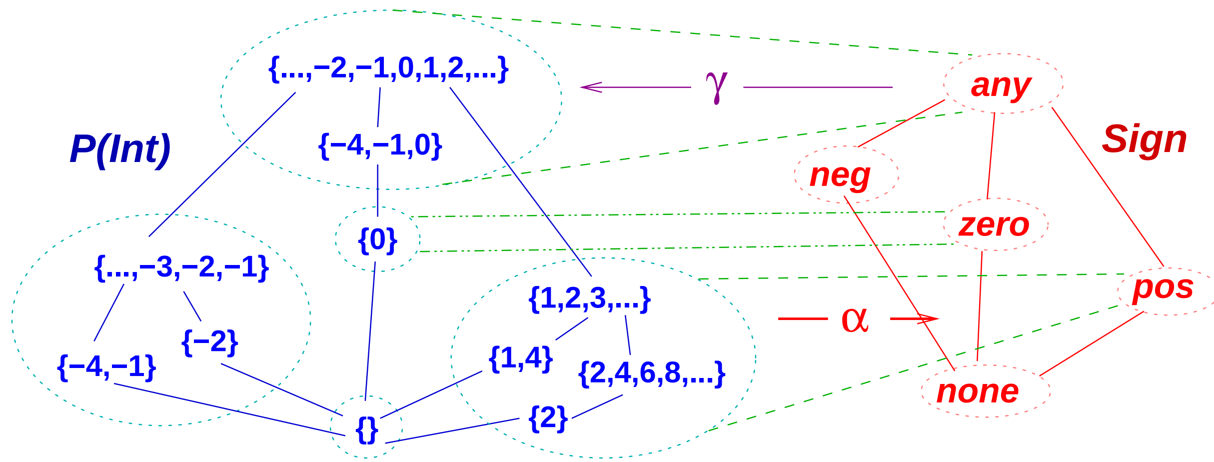
Example:

```
read(x)
if isPositive(x) :
    x:= pred(x)
x:= succ(x)
write(x)
```

Q: Is output pos?

A: abstractly interpret Int by

$Sign = \{neg, zero, pos, any, none\}$:



Standard, collecting interpretation:

$f : \mathcal{P}(Int) \to \mathcal{P}(Int)$:

$\mathrm{isPos}(S) = \{n \in S \mid n > 0\}$

$\mathrm{pred}(S) = \{n - 1 \mid n \in S\}$

$\mathrm{succ}(S) = \{n + 1 \mid n \in S\}$

Abstract interpretation: $f^\sharp : A \to A$:

$\mathrm{isPos}^\sharp(pos) = pos$

$\mathrm{isPos}^\sharp(neg) = none$

$\mathrm{isPos}^\sharp(any) = pos$, etc.

$\mathrm{succ}^\sharp(pos) = pos$

$\mathrm{succ}^\sharp(zero) = pos$

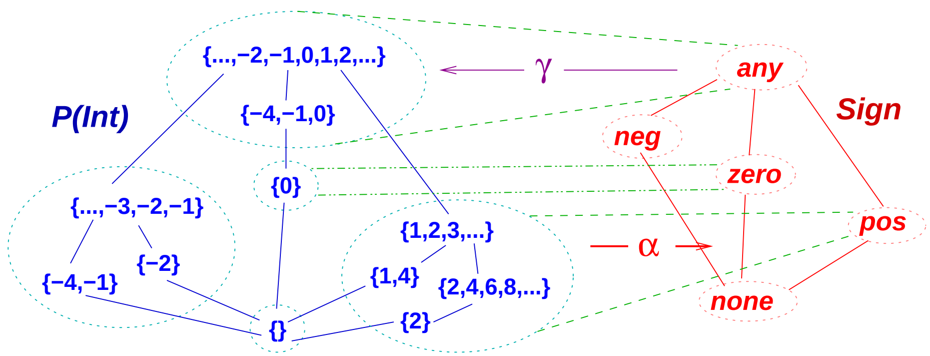
$\mathrm{succ}^\sharp(neg) = any$, etc.

$\mathrm{pred}^\sharp(neg) = neg$

$\mathrm{pred}^\sharp(zero) = neg$

$\mathrm{pred}^\sharp(pos) = any$, etc.

Abstract values = logical properties



Read computational values like $neg \in Sign$ as logical propositions, “isNegative”, etc.

For $S \subseteq \Sigma$, $a, a' \in A$, $\gamma : A \to \mathcal{P}(\Sigma)$, define

- $S \models a$ iff $S \subseteq \gamma(a)$ e.g., $\{-3, -1\} \models neg$
- $a \models a'$ iff $\gamma(a) \subseteq \gamma(a')$ e.g., $neg \models any$
- $a \vdash a'$ iff $a \sqsubseteq a'$ e.g., $neg \vdash any$

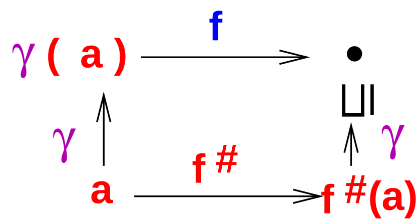
Proposition: *(soundness)* $a \vdash a'$ implies $a \models a'$.

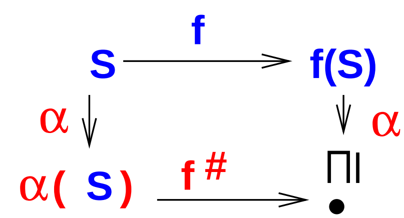
Proposition: *(completeness)* if γ is an upper adjoint of a Galois connection and is 1-1, then $a \models a'$ implies $a \vdash a'$.

Abstract transformers compute on properties

For $\mathsf{f} : \mathrm{PC} \to \mathrm{PC}$, $\mathsf{f}^\sharp : A \to A$ is *sound* iff

$$\mathsf{f} \circ \gamma \sqsubseteq \gamma \circ \mathsf{f}^\sharp \quad \text{iff} \quad \alpha \circ \mathsf{f} \sqsubseteq \mathsf{f}^\sharp \circ \alpha$$





This makes $\mathsf{f}^\sharp$ a *postcondition transformer*:

Proposition: *(soundness)* $S \models a$ implies $\mathsf{f}(S) \models \mathsf{f}^\sharp(a)$.

Example: For, $succ : \mathcal{P}(\mathrm{Int}) \to \mathcal{P}(\mathrm{Int})$, we have $succ\{0\} = \{1\}$, which is soundly mimicked by $succ^\sharp(zero) = pos$.

$\mathsf{f}^\sharp_{\mathrm{best}} = \alpha \circ \mathsf{f} \circ \gamma$ is the *strongest postcondition* transformer for A.

Definition: $\mathsf{f}^\sharp$ is γ*-complete (forwards complete) for* f iff $\mathsf{f} \circ \gamma = \gamma \circ \mathsf{f}^\sharp$ [Giacobazzi01] . $\mathsf{f}^\sharp$ is α*-complete (backwards complete) for* f iff $\alpha \circ \mathsf{f} = \mathsf{f}^\sharp \circ \alpha$ [Cousots00] .

A *has an* internal logic *that* γ *preserves*

First, treat all $a \in A$ as primitive propositions (*isNeg*, *isPos*, etc.).

A *has conjunction* when

$$S \models \phi_1 \sqcap \phi_2 \text{ iff } S \models \phi_1 \text{ and } S \models \phi_2, \text{ for all } S \subseteq \Sigma.$$

That is, $\gamma(\phi \sqcap \psi) = \gamma(\phi) \cap \gamma(\psi)$, for all $\phi, \psi \in A$.

Proposition: When $\gamma : A \to \mathcal{P}(\Sigma)$ is an upper adjoint, then A has conjunction.

$Sign$ has conjunction; so do all predicate-abstraction analyses.

Proposition: When $\gamma(\phi \sqcup \psi) = \gamma(\phi) \cup \gamma(\psi)$, then A has *disjunction*: $S \models \phi \sqcup \psi$ iff $S \models \phi$ or $S \models \psi$.

$Sign$ lacks disjunction: $zero \models neg \sqcup pos$ (because $neg \sqcup pos = any$ but $zero \not\models neg$ and $zero \not\models pos$).

Complete lattice A is *distributive* if $a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$, for all $a, b, c \in A$. When $\sqcap$ is Scott-continuous, then

$$\phi \Rightarrow \psi \;\equiv\; \bigsqcup\{a \in A \mid a \sqcap \phi \sqsubseteq \psi\}$$

satisfies the property, $a \vdash \phi \Rightarrow \psi$ iff $a \sqcap \phi \vdash \psi$.

Proposition: If A is a distributive complete lattice, $\sqcap$ is Scott-continuous, and upper adjoint γ is 1-1, then A has *Heyting implication*, $\phi \Rightarrow \psi$, such that

$$S \models \phi \Rightarrow \psi \text{ iff } \gamma(\alpha(S)) \cap \gamma(\phi) \subseteq \gamma(\psi).$$

That is, $\gamma(\phi \Rightarrow \psi) = \bigcup\{S \in \gamma[A] \mid S \cap \gamma(\phi) \subseteq \gamma(\psi)\}$.

Heyting implication is weaker than classical implication, where $S \models \phi \Rightarrow \psi$ iff $S \cap \gamma(\phi) \subseteq \gamma(\psi)$ iff for all $c \in S$, if $\{c\} \models \phi$, then $\{c\} \models \psi$.

The POS domain for groundness analysis of logic programs uses Heyting implication [Cortesi91,Marriott93] .

If $\gamma(\bot_A) = \emptyset \in \mathcal{P}(\Sigma)$, we have falsity ($\bot$); this yields the logic,

$$\phi \ ::= \ a \mid \phi_1 \sqcap \phi_2 \mid \phi_1 \sqcup \phi_2 \mid \phi_1 \Rightarrow \phi_2 \mid \bot$$

In particular, $\neg\phi$ abbreviates $\phi \Rightarrow \bot$ and defines the *refutation* of ϕ within A, as done in TVLA [Sagiv02] .

$\gamma : A \rightarrow \mathcal{P}(\Sigma)$ is the interpretation function for the internal logic:

$$\gamma(a) = \text{given}$$

$$\gamma(\phi \sqcap \psi) = \gamma(\phi) \cap \gamma(\psi)$$

$$\gamma(\phi \sqcup \psi) = \gamma(\phi) \cup \gamma(\psi)$$

$$\gamma(\phi \Rightarrow \psi) = \bigcup\{S \in \gamma[A] \mid S \cap \gamma(\phi) \subseteq \gamma(\psi)\}$$

$$\gamma(\bot) = \emptyset$$

γ-completeness characterizes the internal logic

The previous interpretation, e.g., for conjunction:

$$\gamma(\phi \sqcap \psi) = \gamma(\phi) \cap \gamma(\psi)$$

shows that γ-completeness is *exactly* the criterion for determining which connectives are embedded in A's internal logic:

Proposition: For $f : \mathcal{P}(\Sigma) \times \mathcal{P}(\Sigma) \times \cdots \to \mathcal{P}(\Sigma)$, A has connective $f^\sharp$ iff $f^\sharp$ is γ-complete for f:

$$\gamma(f^\sharp(\phi_1, \phi_2, \cdots)) = f(\gamma(\phi_1), \gamma(\phi_2), \cdots).$$

Example: For $Sign$, $\mathtt{negate}^\sharp$ is γ-complete for $\mathtt{negate}(S) = \{-n \mid n \in S\}$ (where $\mathtt{negate}^\sharp(pos) = neg$, $\mathtt{negate}^\sharp(neg) = pos$, etc.):

$$\phi ::= a \mid \phi_1 \sqcap \phi_2 \mid \mathtt{negate}^\sharp(\phi)$$

We can state "negate" assertions, e.g., $pos \models \mathtt{negate}^\sharp(neg \sqcap any)$.

Transition functions in the logic: predicate transformers

It is useful to know when $f(S) \models \phi$, *that is,* $S \models [f]\phi$.

Define $[\cdot]$ in terms of $\widetilde{pre}$:

$$[f](S) = \widetilde{pre}_f(S) = \bigcup\{S' \in \Sigma \mid f(S') \subseteq S\}, \qquad \text{for } S \subseteq \mathcal{P}(\Sigma)$$
$$[f^\sharp](a) = \widetilde{pre}_{f^\sharp}(a) = \{a' \in A \mid f^\sharp(a') \sqsubseteq a\}, \qquad \text{for } a \in A$$

When $f^\sharp$ is sound for f, then $\widetilde{pre}_{f^\sharp}$ is sound for $\widetilde{pre}_f$.

Proposition: Assume that γ is an upper adjoint and preserves joins. Then, $\widetilde{pre}_{f^\sharp}$ is γ*-complete* for $\widetilde{pre}_f$ iff $f^\sharp$ is α*-complete* for f.

In this case, we have that $[f^\sharp]$ exists in A's internal logic, where

$$\gamma([f^\sharp]\phi) = \widetilde{pre}_f(\gamma(\phi))$$

But, it is not so common that $f^\sharp$ is α-complete for f.

Logics not internal to the abstract domain

It is common to work with a logic "external" from A's internal logic (e.g., because the transition functions, $f^\sharp : A \to A$, lack α-completeness).

Example: We want this logic for reasoning about $Sign$:

$\phi ::= a \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid [f]\phi$ for $a \in Sign$ and $f \in \{\mathrm{succ}, \mathrm{pred}\}$

where $[\![\cdot]\!] : \mathcal{L} \to \mathcal{P}(\Sigma)$ is defined

$$[\![a]\!] = \gamma(a) \qquad [\![\phi_1 \wedge \phi_2]\!] = [\![\phi_1]\!] \cap [\![\phi_2]\!]$$
$$[\![[f]\phi]\!] = \widetilde{pre}_f[\![\phi]\!] \qquad [\![\phi_1 \vee \phi_2]\!] = [\![\phi_1]\!] \cup [\![\phi_2]\!]$$

But this logic is not internal to $Sign$ (which lacks disjunction, and both $\mathrm{succ}^\sharp$ and $\mathrm{pred}^\sharp$ are not α-complete). *What do we do?*

We can fashion an **external logic**

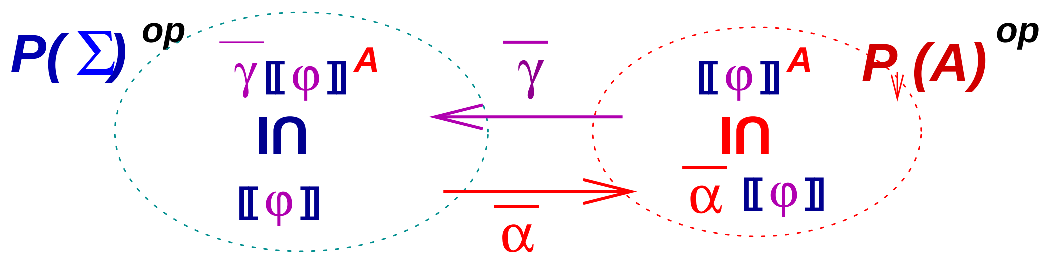
For each $[\![\phi]\!] \subseteq \Sigma$, define

$$[\![\phi]\!]^{\mathcal{A}} = \{a \in A \mid \gamma(a) \subseteq [\![\phi]\!]\}$$

Then, assert $a \vdash \phi$ iff $a \in [\![\phi]\!]^{\mathcal{A}}$.

This definition follows from a Galois connection whose abstract domain is $\mathcal{P}_{\downarrow}(A)^{\mathrm{op}}$ — downclosed subsets of A, ordered by superset:

$$\overline{\gamma}(T) = \bigcup\{\gamma(a) \mid a \in T\}$$
$$\overline{\alpha}(S) = \{a \mid \gamma(a) \subseteq S\}$$



That is, $[\![\phi]\!]^{A} = \overline{\alpha}[\![\phi]\!]$.

The inverted ordering gives *underapproximation*: $[\![\phi]\!] \supseteq \overline{\gamma}([\![\phi]\!]^{\mathcal{A}})$. This form of external logic is standard in "abstract model checking."

It is also standard to write an inductively defined approximation to $\overline{\alpha}[\![\phi]\!]$:

$$
\begin{aligned}
[\![a]\!]^{\mathcal{A}}_{\mathrm{ind}} &= \overline{\alpha}(\gamma(a)) \\
[\![\phi_1 \wedge \phi_2]\!]^{\mathcal{A}}_{\mathrm{ind}} &= [\![\phi_1]\!]^{\mathcal{A}}_{\mathrm{ind}} \cap [\![\phi_2]\!]^{\mathcal{A}}_{\mathrm{ind}} \\
[\![\phi_1 \vee \phi_2]\!]^{\mathcal{A}}_{\mathrm{ind}} &= [\![\phi_1]\!]^{\mathcal{A}}_{\mathrm{ind}} \cup [\![\phi_2]\!]^{\mathcal{A}}_{\mathrm{ind}} \\
[\![[f]\phi]\!]^{\mathcal{A}}_{\mathrm{ind}} &= \widetilde{pre}_{f^\sharp}[\![\phi]\!]^{\mathcal{A}}_{\mathrm{ind}} = \{a \in A \mid f^\sharp(a) \in [\![\phi]\!]^{\mathcal{A}}_{\mathrm{ind}}\}
\end{aligned}
$$

Entailment and provability are as expected: $a \models \phi$ iff $\gamma(a) \subseteq [\![\phi]\!]$, and $a \vdash \phi$ iff $a \in [\![\phi]\!]^{\mathcal{A}}_{\mathrm{ind}}$.

Soundness ($\vdash$ implies $\models$) is immediate, and completeness ($\models$ implies $\vdash$) follows when $\overline{\alpha} \circ [\![\,\cdot\,]\!] = [\![\,\cdot\,]\!]^{\mathcal{A}}_{\mathrm{ind}}$. This is called *logical best preservation* or *logical $\overline{\alpha}$-completeness* [Cousots00,Schmidt06] .

Embedding *the external logic within the internal*

Say that γ is an upper adjoint and that it *preserves joins*, that is,

$$\overline{\gamma}T = \bigcup_{a \in T} \gamma(a) = \gamma(\bigsqcup_{a \in T} a) = \gamma(\sqcup T)$$

So, $\overline{\gamma}[\mathcal{P}_{\downarrow}(A)] = \gamma[A]$ — their ranges are equal — and *there is no new expressivity gained by using sets of A-elements.*

Proposition: If A is a complete lattice and $\gamma : A \longrightarrow \mathcal{P}(\Sigma)$ preserves joins (as unions) *and* meets (as intersections), then

- γ is the upper adjoint of an *overapproximating* Galois connection between $(\mathcal{P}(\Sigma), \subseteq)$ and $(A, \sqsubseteq)$, where $\alpha_o(S) = \sqcap\{a \mid S \subseteq \gamma(a)\}$.

- γ is the upper adjoint of an *underapproximating* Galois connection between $(\mathcal{P}(\Sigma), \supseteq)$ and $(A, \sqsupseteq)$, where $\alpha_u(S) = \sqcup\{a \mid S \supseteq \gamma(a)\}$.

1. For state-transition functions, $f : \mathcal{P}(\Sigma) \rightarrow \mathcal{P}(\Sigma)$, apply the first Galois connection, giving the *computational interpretation* of f: $f^{\sharp}_{best} = \alpha_o \circ f \circ \gamma$.

2. For logical connectives, $[\![\mathtt{f}(\phi_1, \phi_2, \cdots)]\!] = f([\![\phi_1]\!], [\![\phi_2]\!], \cdots)$, apply the second Galois connection, for the *logical interpretation* of f: $f^{\flat}_{best} = \alpha_u \circ f \circ (\gamma \times \gamma \times ...)$ and $[\![\mathtt{f}(\phi_1, \phi_2, \cdots)]\!]^{\mathcal{A}}_{ind} = f^{\flat}_{best}([\![\phi_1]\!]^{\mathcal{A}}_{ind}, [\![\phi_2]\!]^{\mathcal{A}}_{ind}, \cdots)$,

When the $f^{\flat}_{best}$s are α_u-complete, then $[\![\cdot]\!]^{\mathcal{A}}_{ind} = \alpha_u \circ [\![\cdot]\!]$, and the resulting internal logic *proves the same assertions* as the external logic:

- First, for $[\![\phi]\!]^{\mathcal{A}} = \overline{\alpha}[\![\phi]\!] \in \mathcal{P}_{\downarrow}(A)$, recall that $a \vdash \phi$ iff $a \in [\![\phi]\!]^{\mathcal{A}}$.
- Next, for $[\![\phi]\!]^{\mathcal{A}}_{ind} = \alpha_u[\![\phi]\!] \in A$, define $a \vdash \phi$ iff $a \sqsubseteq [\![\phi]\!]^{\mathcal{A}}_{ind}$.

Theorem: For all $a \in A$, $a \in [\![\phi]\!]^{\mathcal{A}}$ iff $a \sqsubseteq [\![\phi]\!]^{\mathcal{A}}_{ind}$.

Conclusions

- A static analysis is "logical" in that it *computes proofs* (via $\sqsubseteq$) in the abstract domain, A, that are *sound* (via $\models$, i.e., γ) in the concrete domain, Σ.

- γ-*completeness* (homomorphism property) characterizes the *internal logic* one can soundly validate on A-values, using $\sqsubseteq$. Assertions not in the internal logic can be approximated within an *external* logic defined with sets of A-values and checked using $\in$.

- When γ preserves joins and meets from A to $\mathcal{P}(\Sigma)$, the external logic can be *embedded* within the abstract domain, letting it *overapproximate* computations on Σ and *underapproximate* assertions on Σ.

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