Underapproximation in VMCAI

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Belated 60th birthday greetings to Ed Clarke and a big thank you for your leadership in the field!

90% of this talk originates from...

1. *Clarke, Grumberg, Long [POPL92, TOPLAS 94]:* state-space abstractions via semi-homorphisms to validate $\forall$CTL*-properties.

2. *Cousot-Cousot [POPL77, POPL79, JLC 92]:* abstract-interpretation frameworks that synthesize abstract functions and ensure soundness

3. *Dams [thesis 96, and TOPLAS 97 with Gerth and Grumberg]:* formalization of underapproximation functions on abstract models

Introduction

Overapproximation *"states the possibilities" or "covers the behaviors" of an entity or process*

Examples:

- *a program's control-flow graph* — its paths represent a superset of the program's actual executions
- *an entity-relation model* — it states all legal relationships between objects in a knowledge base
- *data-type information* of a program's variables states the variables' range of values
- *logical program assertions* that have not been refuted state "what the program might possibly do"

Underapproximation *"lists the necessities" or gives concrete examples of an entity or process*

Examples:

- *logical program assertions* that have been proved true or are required of a program define a subset of the program's theory
- *test-execution traces* assert guaranteed program behaviors
- *execution monitoring* remembers values that have been assigned to program variables at some point
- *an object subdiagram* displays relationships the program must construct during its execution

Testing is a commonly used underapproximation of program behavior; so is *Bounded Model Checking* (trace generation to some fixed $k \geq 0$).

Such *concrete underapproximations* construct *witnesses* to an *existential property* (e.g., a trace that ends in an error state).

Sometimes, concrete underapproximations can be clevely applied to other ends:

- **Păsăreanu, et al. CAV05:** each concrete trace refines a predicate-abstraction model of the program's control-flow graph; the limit of the refinements is a model that is bisimular to the program's concrete control-flow graph (modulo the predicates selected for the analysis)

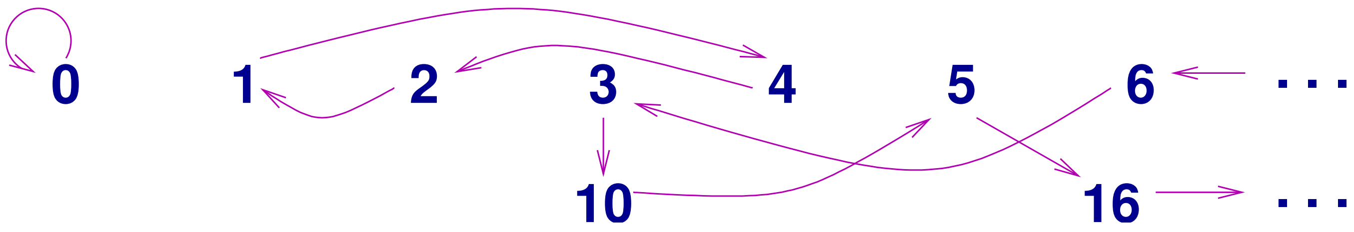
- **Grumberg, et al. POPL05**: SAT-generated *BMC* traces attempt to refute a property via a counterexample trace. If no counterexample found, the SAT-trace-proof is disassembled to see if the bounding on the traces appears in the trace-proof. If no, then the trace-proof is *rebuilt* into a proof that *proves* the property.

We will study over- and underapproximations calculated on **abstractions** *of states/data*

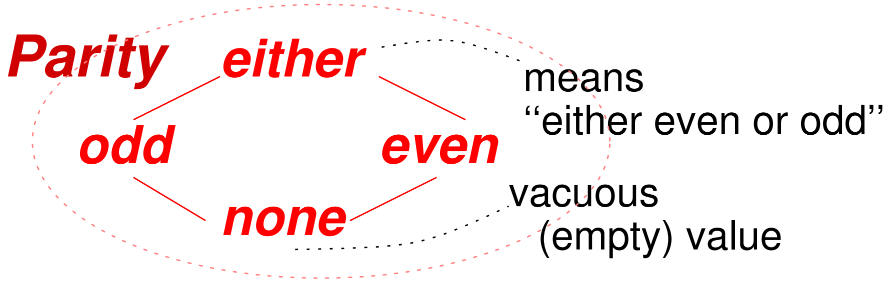
Example: the Collatz function:

```
loop
  n div2 == 0 ?  n := n div2;
  n div2 == 1 ?  n := 3n + 1;
endloop
```

The function's graph — what really happens:



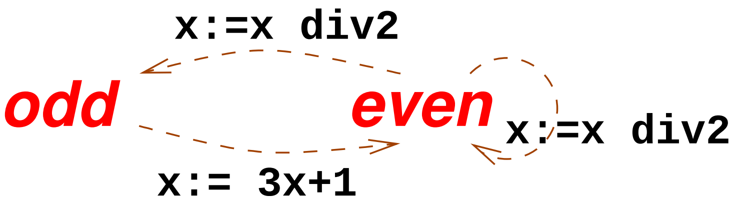
The Parity abstraction of the function's domain:



Example: $2 \in \gamma(\mathrm{even}) = \{0, 2, 4, \cdots\}$, $3 \in \gamma(\mathrm{odd}) = \{1, 3, 5, \cdots\}$, for the modelling (*concretization*) function, $\gamma : \mathrm{Parity} \to \mathcal{P}(\mathrm{Nat})$.

An overapproximation (“may”) graph based on Parity

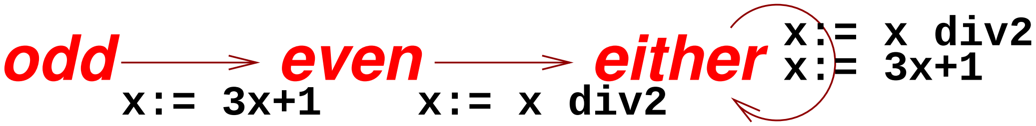
(∃∃-*approximation*): covers all concrete paths but contains “false paths”:



Example: $8 \rightarrow 4$ and $10 \rightarrow 5$, hence $even \rightarrow even$ and $even \rightarrow odd$. **You may add transitions to this model and it remains an overapproximation.**

An underapproximation (”must”) graph based on Parity

(∀∃-*approximation*): all paths are guaranteed to exist as concrete executions:



Example: for every odd number, $2n + 1 \geq 0$, there is some $2m$ such that $2n + 1 \rightarrow 2m$, hence $odd \rightarrow even$. **You may remove transitions from this model and it remains an underapproximation.**

We validate logical properties on abstract models

Let C be the set of concrete values (states) and A be the set of abstract ones.

Say that $\phi \in \mathcal{L}$*, a logic, and* $[\![\phi]\!] \subseteq C$*.*

For $c \in C$*, write* $c \models \phi$ *when* $c \in [\![\phi]\!]$*.*

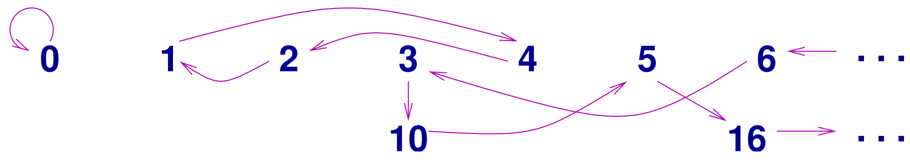
For $a \in A$, we wish to check $a \models^{A} \phi$ and *infer* $c \models \phi$ for those $c \in \gamma(a)$. Recall that $\gamma : A \to \mathcal{P}(C)$ is the modelling/concretization function.

Slogan: **Overapproximate** *the computation and* **underapproximate** *the logic*:

if $a \models^{A} \phi$
(i.e., $a \in [\![\phi]\!]^{A}$),
then, for all $c \in \gamma(a)$
$c \models \phi$ (i.e, $c \in [\![\phi]\!]$)

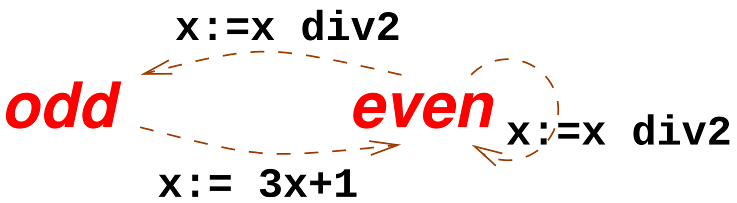
$$
\begin{tikzcd}
\mathbf{a} \arrow[d, "\gamma"'] & \in & {[\![\,\varphi\,]\!]^{A}} \arrow[d, "\overline{\gamma}"] \\
\gamma(\mathbf{a}) \arrow[d, phantom, "\rotatebox{90}{$\in$}", "\text{(overappr)}"'] & \subseteq & \overline{\gamma}\,{[\![\,\varphi\,]\!]^{A}} \arrow[d, phantom, "\rotatebox{90}{$\subseteq$}", "\text{(underappr)}"] \\
\mathbf{c} & \in & {[\![\,\varphi\,]\!]}
\end{tikzcd}
$$

But we also know that an *over* approximating model is used to validate *universal* properties and an *under* approximating model is used to validate *existential* properties:

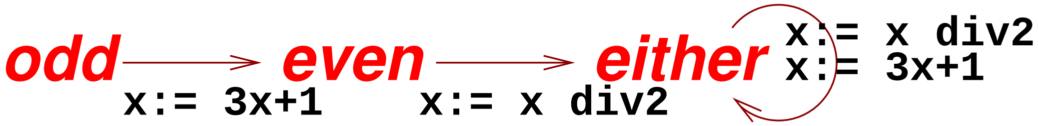



```
loop
  n div2 == 0?  n:= n div2;
  n div2 == 1?  n:= 3n+1;
endloop
```

$$odd \models^{A} \Box even$$
$$even \models^{A} \forall(\mathrm{G}even)$$



$$odd \models^{A} \Diamond even$$
$$odd \models^{A} \exists(\mathrm{F}even)$$



We might even *mix* the two: $even \models^{A} \Box(even \vee \Diamond even)$

How do we make sense of this?

How we make sense of over-underapproximation

1. Kripke structures, simulations, Galois connections
2. Logics
3. How to underapproximate a logic
4. How to over- and underapproximate a program's control structure
5. How to underapproximate a data structure
6. Over- and underapproximation in specification

Kripke structures, simulations, Galois connections

A program's semantics is coded as a State Transition System

Definition: An *STS* is (C, f), where C is the state set and $f \subseteq C \times C$ (or $f : C \to \mathcal{P}(C)$) is the transition relation (function).

Collatz example: C is Nat, and

$$f = \{(n, n\ div2) \mid n\ div2 == 0\} \cup \{(n, 3n+1) \mid n\ div2 == 1\}$$

A *run* (*trace*) is $c_0 \to c_1 \to \cdots c_i \to c_{i+1} \to \cdots$ such that for all $i \geq 0$, $(c_i, c_{i+1}) \in f$.

Let $Prop$ be a set of primitive properties and let $\gamma : Prop \to \mathcal{P}(C)$ interpret it.

Definition: A *Kripke structure* is an STS $+\ \gamma$.

Collatz example: $Prop = \{even, odd\}$, and

$$\gamma(even) = \{2n \mid n \geq 0\}, \quad \gamma(odd) = \{2n+1 \mid n \geq 0\}$$

When (C, f) *is "too large," we must* **abstract** *it*

Say that $\gamma : \mathrm{Prop} \to C$ *partitions* C $(\forall c \in C, \exists! p \in \mathrm{Prop}, c \in \gamma(p))$.

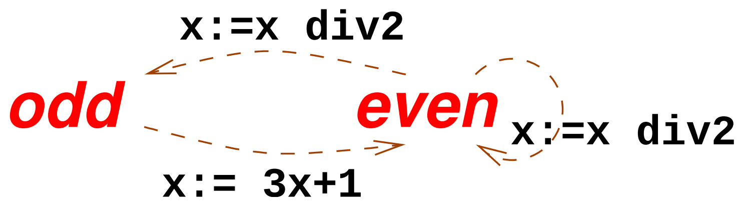
Then we can define the abstract STS, $(\mathrm{Prop}, f^\sharp)$, where

$$(p, q) \in f^\sharp \text{ iff } \exists c \in \gamma(p), \exists c' \in \gamma(q), (c, c') \in f$$

$f^\sharp$ *is an* $\exists\exists$*-relation.*

Collatz example:

$$\mathrm{Prop} = \{even, odd\},$$
$$\gamma(even) = \{2n \mid n \geq 0\}, \quad \gamma(odd) = \{2n + 1 \mid n \geq 0\}, \text{ and}$$
$$f^\sharp = \{(odd, even), (even, even), (even, odd)\}$$



Soundness of the abstraction is asserted by a simulation

Define the *modelling relation*, $\rho \subseteq C \times A$, from γ as

$$c\ \rho\ p \textit{ iff } c \in \gamma(p).$$

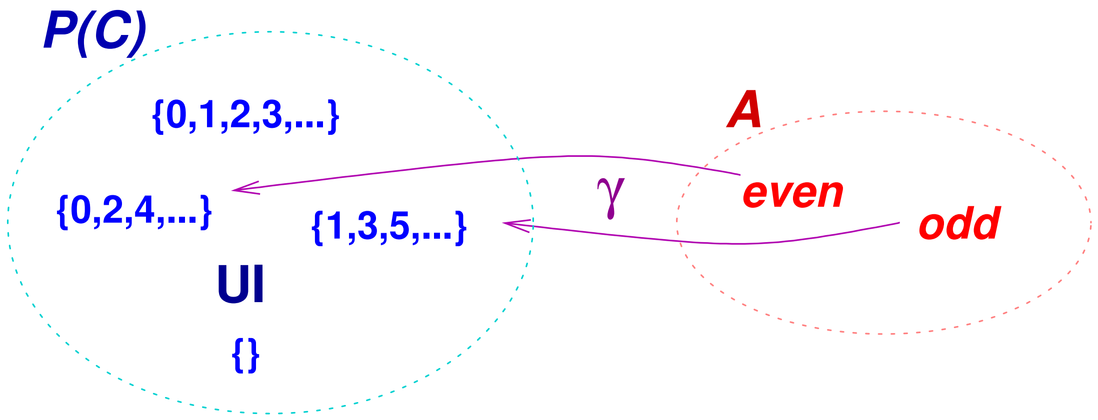
Definition: For STSs (C, f) and $(A, f^\sharp)$, ρ is a *simulation* iff for all $c \in C$, $a \in A$,

$$c\ \rho\ a \text{ and } (c, c') \in f \text{ imply there exists } a' \in A \text{ such that } (a, a') \in f^\sharp \text{ and } c'\ \rho\ a'.$$

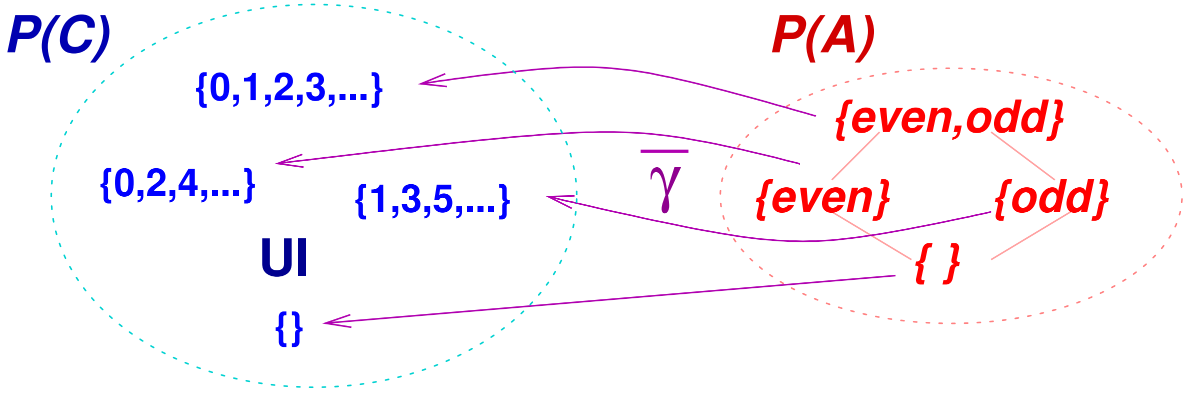
$f^\sharp$ mimicks f, modulo ρ. We write $f \lhd f^\sharp$:

$$\begin{tikzcd}
c_0 \arrow[r, "f"] \arrow[d, phantom, "\in" description] & c_1 \arrow[r, "f"] \arrow[d, phantom, "\in" description] & \cdots \arrow[r, "f"] & c_i \arrow[r, "f"] \arrow[d, phantom, "\in" description] & c_{i+1} \arrow[r] \arrow[d, phantom, "\in" description] & \cdots \\
\gamma(a_0) \arrow[d, dotted, no head, "\rho"'] & \gamma(a_1) \arrow[d, dotted, no head, "\rho"'] & & \gamma(a_i) \arrow[d, dotted, no head, "\rho"'] & \gamma(a_{i+1}) \arrow[d, dotted, no head, "\rho"'] & \\
a_0 \arrow[r, "f^\#"'] & a_1 \arrow[r, "f^\#"'] & \cdots \arrow[r, "f^\#"'] & a_i \arrow[r, "f^\#"'] & a_{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

Given γ:



we lift it to $\overline{\gamma} : \mathcal{P}(A) \rightarrow \mathcal{P}(C)$:



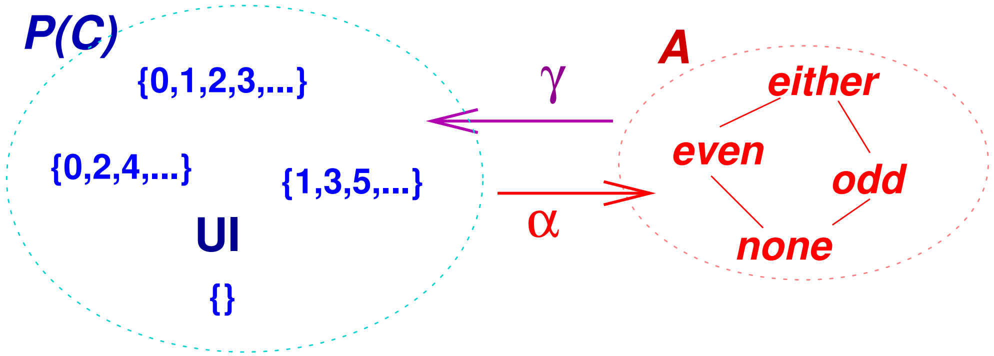
$\overline{\gamma}(T) = \cup_{a \in T}\, \gamma(a)$

This lets us define an abstract transition *function*, $f^\sharp : A \rightarrow \mathcal{P}(A)$.

Collatz example: $f^\sharp(even) = \{even, odd\}$, $f^\sharp(odd) = \{even\}$.

Theorem: *Simulation equals abstract-interpretation soundness:*
For $f : C \rightarrow \mathcal{P}(C)$, $f^\sharp : A \rightarrow \mathcal{P}(A)$, $f \lhd f^\sharp$ iff $f^* \circ \gamma \sqsubseteq \gamma \circ f^\sharp$, where
$f^*(S) = \cup_{c \in S} f(c)$.

The previous slide hints that γ and $\overline{\gamma}$ have inverse maps — *they do* — and this situation is called a *Galois connection*:

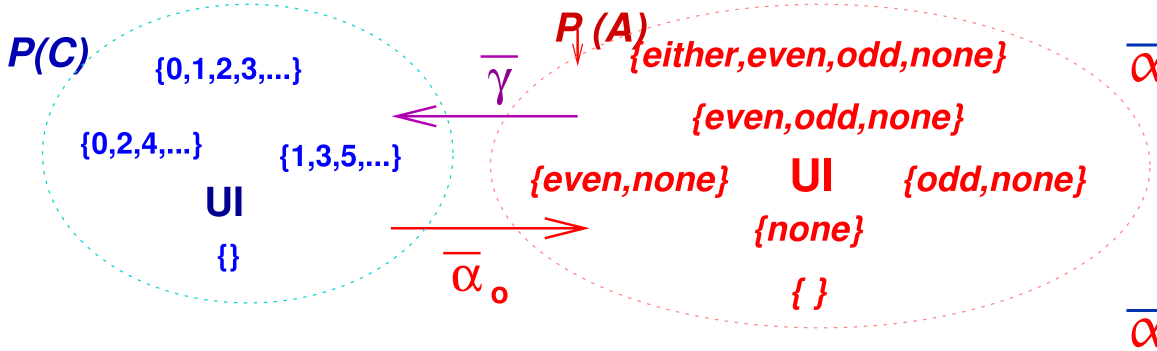


$$\alpha(S) = \sqcap\{a \mid \gamma(a) \subseteq S\}$$

e.g., $\alpha\{2,6\} = \mathit{even}$

$\alpha\{0,1,2,3\} = \mathit{either}$

either and none help Parity become a complete lattice.



$$\overline{\alpha}_o(S) = \cap\{T \mid S \subseteq \overline{\gamma}(T)\}$$

e.g.,

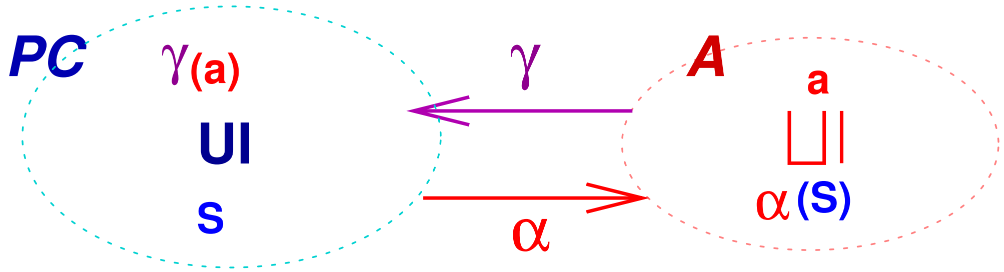
$\overline{\alpha}_o\{2,6\} = \{\mathit{even}, \mathit{none}\}$

$\overline{\alpha}_o\{0,2,3\} = \{\mathit{even},\mathit{odd},\mathit{none}\}$

Down-closed sets are needed to make $\overline{\alpha}_o$ monotone.

Note that $\overline{\gamma}\{\mathit{either}, \mathit{even}, \mathit{odd}, \mathit{none}\} = \overline{\gamma}\{\mathit{even}, \mathit{odd}, \mathit{none}\}$; the former is superfluous and can be deleted. The same is true for $\{\}$ and $\{\mathit{none}\}$.

Why are Galois connections important?



$\alpha(\gamma(a)) \sqsubseteq a$

$S \subseteq \gamma(\alpha(S))$

1. for $S \in PC$, $\alpha(S)$ gives the most precise approximant in A
2. it formalizes (over)approximation ($\subseteq$ in PC above!): $S \subseteq \gamma(\alpha(S))$, and for all $S \in PC$, $f(S) \subseteq \gamma(f^\sharp(\alpha(S)))$
3. it ensures that $\sqcap_A$ is *conjunction* — $a \sqcap a'$ is read as $a \wedge a'$ — because $\gamma(\sqcap_i a_i) = \cap_i \gamma(a_i)$
4. for $f : PC \rightarrow PC$, we can synthesize the most precise approximation, $f^\sharp_{best} = \alpha \circ f \circ \gamma$

   Dams: for $f : C \rightarrow \mathcal{P}(C)$ *in an STS,* $f^\sharp_{best} : A \rightarrow \mathcal{P}_\downarrow(A)$ *is*

   $f^\sharp_{best}(a) = \downarrow\{\alpha\{c'\} \mid c \in \gamma(a), c' \in f(c)\}$ *— the minimal $\exists\exists$-relation*

Logics

We use a logic to state properties of (C, f)

$$p \in \mathrm{Prop} \qquad \phi \in \mathcal{L} \qquad\qquad [\![\cdot]\!] \subseteq C$$

$$\phi \ ::= \ p \mid \mathrm{op}_g^k(\phi_i)_{i<k} \qquad\qquad [\![p]\!] = \gamma(p)$$

$$[\![\mathrm{op}_g^k(\phi_i)_{i<k}]\!] = g[\![\phi_i]\!]_{i<k}$$

Collatz example:

$$\phi \ ::= \ \mathit{even} \mid \mathrm{odd} \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid \Box\phi$$

$$[\![\mathit{even}]\!] = \gamma(\mathit{even}) \qquad\qquad [\![\mathrm{odd}]\!] = \gamma(\mathrm{odd})$$

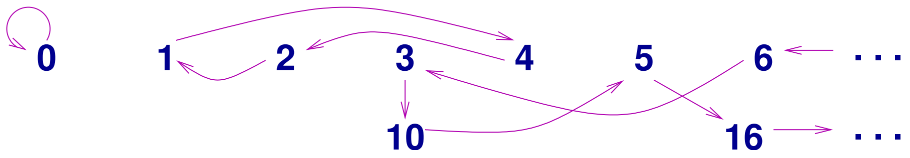
$$[\![\phi_1 \wedge \phi_2]\!] = [\![\phi_1]\!] \cap [\![\phi_2]\!]$$

$$[\![\phi_1 \vee \phi_2]\!] = [\![\phi_1]\!] \cup [\![\phi_2]\!]$$

$$[\![\Box\phi]\!] = \tilde{\mathrm{pre}}_f[\![\phi]\!], \text{ where } \tilde{\mathrm{pre}}_f(S) = \{c \mid f(c) \subseteq S\}$$

$c \in [\![\Box\phi]\!]$ *means "*$\forall f.\phi$*" — all next* $f(c)$*-states belong to* ϕ *— for transition function,* $f : C \to \mathcal{P}(C)$*.*

Collatz examples:



$$\gamma\{\text{even}\} = \{2n \mid n \geq 0\}$$
$$\gamma\{\text{odd}\} = \{2n+1 \mid n \geq 0\}$$

$3 \in [\![\text{odd}]\!]$

$3 \in [\![\text{odd} \wedge \Box \text{even}]\!]$

$10 \in [\![\Box \text{odd}]\!]$

$12 \notin [\![\Box \text{odd}]\!]$

We'll examine path logics, where $[\![\psi]\!] \subseteq C^*$, later.

Approximating the logic

We will apply the logic to the abstract STS, $(A, f^\sharp)$

For $(A, f^\sharp)$, $\gamma : A \to \mathcal{P}(C)$, we must define a $[\![\cdot]\!]^A \subseteq A$ so that it is

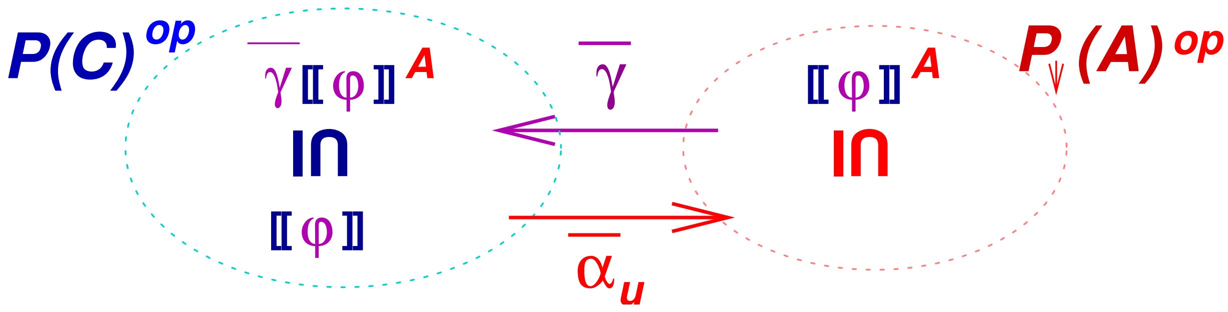
1. ***weakly preserving (sound)***:

   for all $a \in A$, $a \in [\![\phi]\!]^A$ implies $c \in [\![\phi]\!]$, for all $c \in \gamma(a)$

2. ***best preserving***:

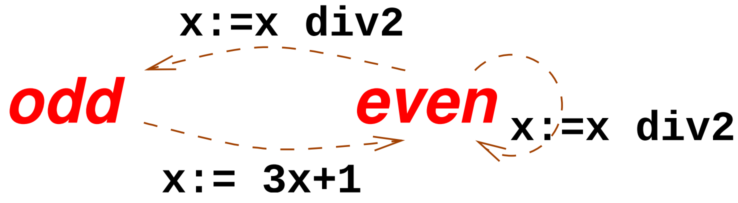
   for all $a \in A$, $a \in [\![\phi]\!]^A$ iff $\gamma(a) \subseteq [\![\phi]\!]$

For soundness, $[\![\cdot]\!]^A$ must *underapproximate* $[\![\cdot]\!]$:



that is, $\overline{\gamma}[\![\phi]\!]^A \subseteq [\![\phi]\!]$.

Collatz examples: abstract transition function, $f^\sharp : A \to \mathcal{P}(A)$:



$$f^\sharp(\mathrm{odd}) = \{\mathit{even}\}$$
$$f^\sharp(\mathit{even}) = \{\mathit{even}, \mathrm{odd}\}$$

We anticipate that

$\mathrm{odd} \in [\![\mathrm{odd}]\!]^A$

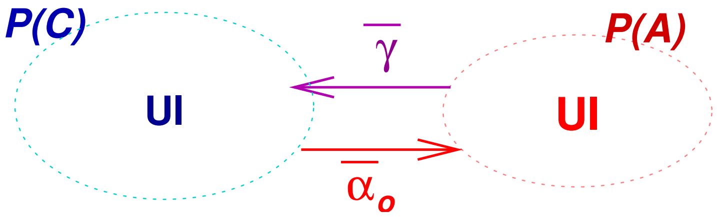
$\mathrm{odd} \in [\![\mathrm{odd} \wedge \Box \mathit{even}]\!]^A$

but $\mathit{even} \notin [\![\Box \mathrm{odd}]\!]^A$ due to loss in precision.

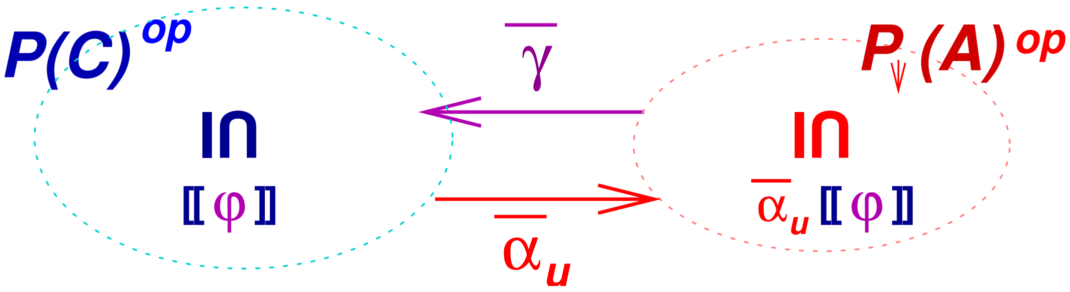
How do we define $[\![\,\cdot\,]\!]^A$?

How to compute $[\![\cdot]\!]^A$ *from* $[\![\cdot]\!]$

For this Galois connection,



$\overline{\gamma} : \mathcal{P}_{\downarrow}(A) \to \mathcal{P}(C)$ preserves $\sqcup$ *as well as* $\sqcap$ — it can be *inverted*:



$\overline{\alpha}_u : \mathcal{P}(C)^{op} \to \mathcal{P}_{\downarrow}(A)^{op}$ is new:

$\overline{\alpha}_u(S) = \{a \mid \gamma(a) \subseteq S\}$

$\overline{\alpha}_u$ tells us how to abstract $[\![\phi]\!]$ — use $\overline{\alpha}_u[\![\phi]\!]$!

Theorem: $\overline{\alpha}_u[\![\cdot]\!]$ is best preserving. But it's not defined inductively....

A sound inductive definition requires a $g^\flat$

$$[\![p]\!]^A = \overline{\alpha}_u[\![p]\!] = \{a \mid \gamma(a) \subseteq \gamma(p)\}$$

$$[\![op_g^k(\phi_i)_{i<k}]\!]^A = g^\flat[\![\phi_i]\!]^A{}_{i<k}$$

where $g^\flat$ *underapproximates* g: $\overline{\gamma}(g^\flat(T)) \subseteq g(\overline{\gamma}(T))$, for $T \in \mathcal{P}_\downarrow(A)$.

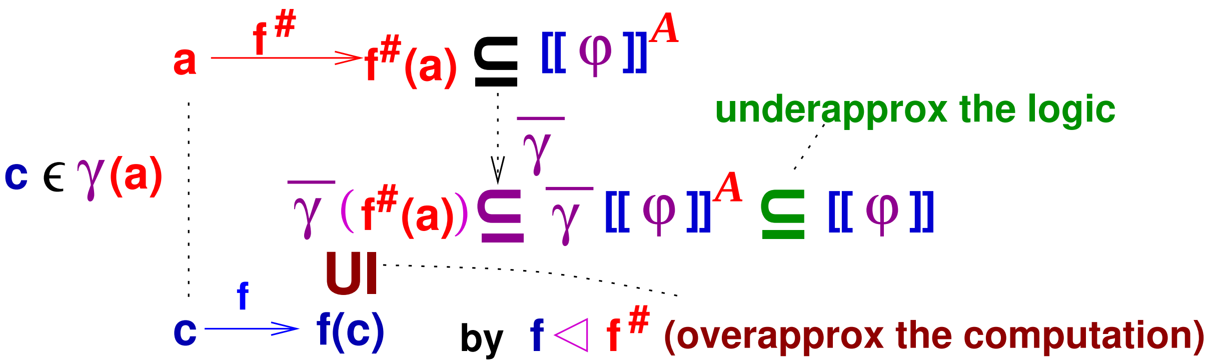
Sometimes, obvious selections for $g^\flat$ work well:

$$[\![\phi_1 \wedge \phi_2]\!]^A = [\![\phi_1]\!]^A \cap [\![\phi_2]\!]^A$$

$$[\![\phi_1 \vee \phi_2]\!]^A = [\![\phi_1]\!]^A \cup [\![\phi_2]\!]^A$$

because $\overline{\gamma}$ preserves both meets and joins in $\mathcal{P}_\downarrow(A)$.

$[\![\Box\phi]\!]^A = \widetilde{pre}_{f^\sharp}[\![\phi]\!]^A$, where $f \lhd f^\sharp$, because



if all $f^\sharp(a)$'s answers have ϕ, so must $f(c)$'s, because $f^\sharp$ "covers" f

The most precise (largest-set) inductive definition

$$[\![p]\!]^{\mathcal{A}} = \overline{\alpha}_u [\![p]\!]$$

$$[\![op^k_g(\phi_i)_{i<k}]\!]^{\mathcal{A}} = g^\flat_{best} [\![\phi_i]\!]^{\mathcal{A}}{}_{i<k},$$

where $g^\flat_{best} = \overline{\alpha}_u \circ g \circ \overline{\gamma}^k$, for $g : \mathcal{P}(C)^k \to \mathcal{P}(C)$

Example: $(\tilde{pre}_f)^\flat_{best} = \overline{\alpha}_u \circ \tilde{pre}_f \circ \overline{\gamma} = \{a \mid f^*(\gamma(a)) \subseteq \overline{\gamma}(T)\}$.

Proposition: (weak preservation) $\overline{\alpha}_u [\![\phi]\!] \supseteq [\![\phi]\!]^{\mathcal{A}}$

Theorem: (best preservation) When $\overline{\gamma}$ is *complete with respect to* g, then $\overline{\alpha}_u [\![\phi]\!] = [\![\phi]\!]^{\mathcal{A}}$.

$\overline{\gamma}$ is *complete w.r.t* g iff soundness is exact: $g \circ \overline{\gamma} = \overline{\gamma} \circ g^\flat_{best}$.

Theorem: $(\tilde{pre}_f)^\flat_{best} = \tilde{pre}_{(f^*)^\sharp_{best}} \sqsubseteq \tilde{pre}_{f^\sharp}$

The preimage of $(f^*)^\sharp_{best} = \overline{\alpha}_o \circ f^* \circ \gamma$, the most precise overapproximation of f, equals the most precise underapproximation of $\tilde{pre}_f$.

An existential assertion: $\Diamond\phi$

$\Box\phi$ is a "universal assertion." In contrast, $\Diamond\phi$ *asserts there exists a next state with* ϕ*:*

$$\begin{aligned} [\![\Diamond\phi]\!] &= pre_f[\![\phi]\!], \\ &\quad \text{where } f : C \to \mathcal{P}(C) \text{ and } pre_f(S) = \{c \mid f(c) \cap S \neq \emptyset\}. \end{aligned}$$

Here is its most precise (largest) underapproximation:

$$\begin{aligned} [\![\Diamond\phi]\!]^A &= (\overline{\alpha}_u \circ pre_f \circ \overline{\gamma})[\![\phi]\!]^A \\ &= \{a \mid \text{for all } c \in \gamma(a), f(c) \cap \overline{\gamma}(T) \neq \emptyset\} \end{aligned}$$

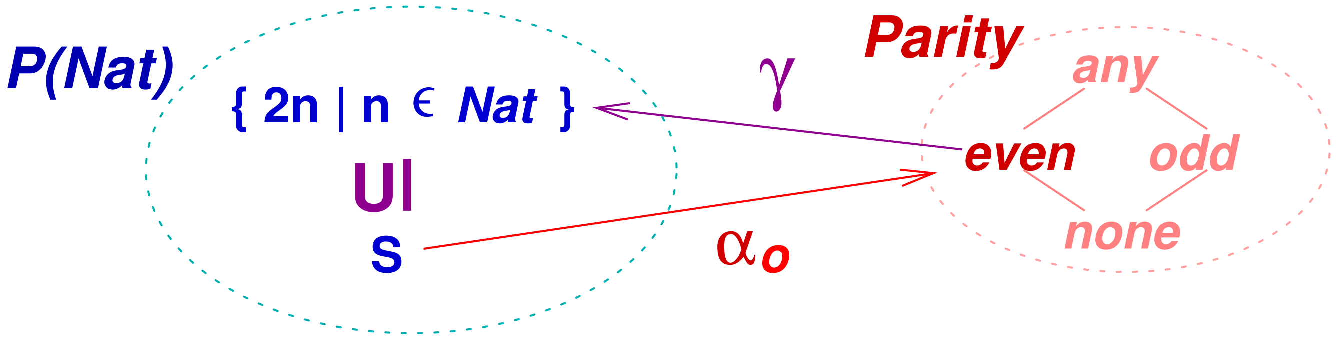
This is a $\forall\exists$-set.

Can we use $[\![\Diamond\phi]\!]^A = pre_{f^\sharp}[\![\phi]\!]^A$ **? NO.** **Collatz example:** $even \to even$, $10 \in \gamma(even)$, yet $10 \to 5$ only — *an even is not guaranteed to transit to an even.*

Can we use $[\![\Diamond\phi]\!]^A = pre_{f_0^\flat}[\![\phi]\!]^A$**, where** $f_0^\flat = \overline{\alpha}_u \circ f^* \circ \gamma$ **? NO.** All sets, $[\![\phi]\!]^A$, are *downwards closed* in A. But $pre_{f_0^\flat}(T)$, for downclosed T, is an *upwards-closed set* !

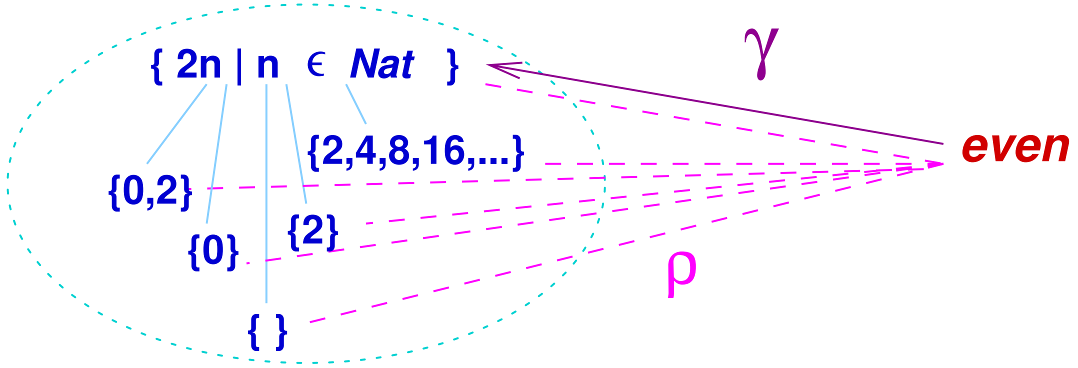
Underapproximating control

Some perspective: Over-approximation *states a property of a program's outputs*



$even \in Parity$ asserts "$\forall even$" — all concrete outputs in set S are even-valued. (*We might write* $S \; \rho \; even$ *or* $S \models even$.)

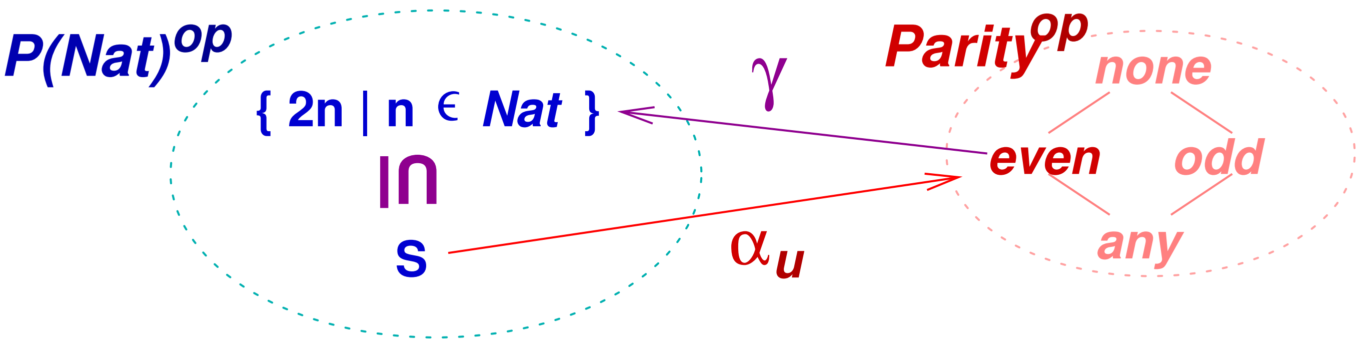
The upper adjoint, γ, selects the largest set approximated by $even$:



$$\gamma(a) = \cup\{S \mid S \; \rho \; a\}$$

Under-approximation *might be stated as the dual*

Here, *even* asserts that all evens are *included* in the concrete outputs:



But as we saw earlier, this subset-underapproximation is not well suited to underapproximating computation:

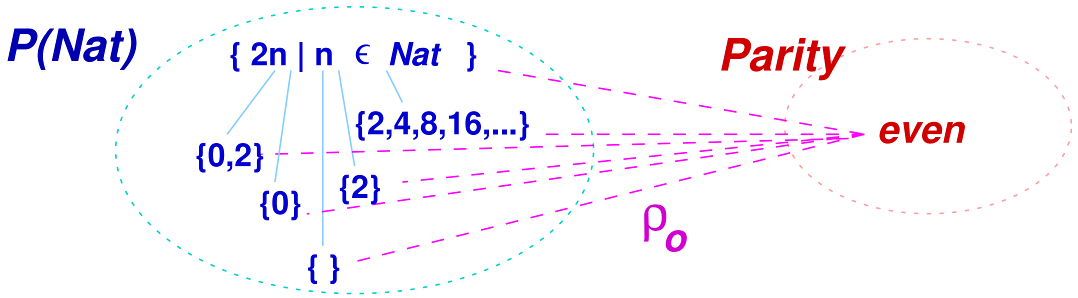
Collatz example: $f = \{0 \to 0, 1 \to 4, 2 \to 1, 3 \to 10, \cdots\}$

For $f_0^\flat = \overline{\alpha}_u \circ f^* \circ \gamma$,

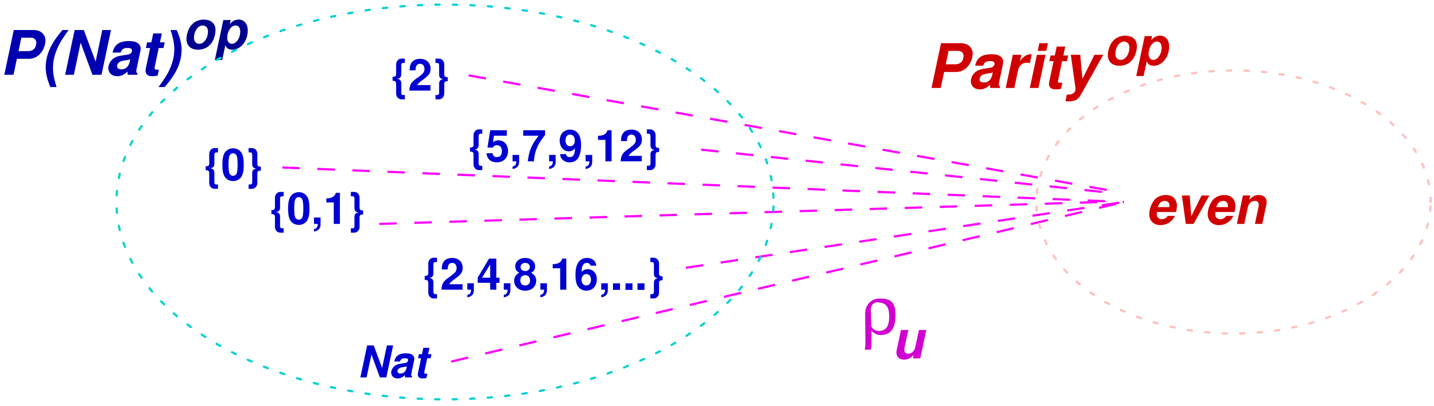
$f_0^\flat(\mathit{odd}) = \overline{\alpha}_u(f^*\{1, 3, 5, \cdots\} = \overline{\alpha}_u\{4, 10, 16, 22, \cdots\} = \{\mathit{none}\}$ (!)

Under-approximation as existential quantification

If the over-approximating $\mathit{even} \in \mathit{Parity}$ asserts "$\forall \mathit{even}$,"

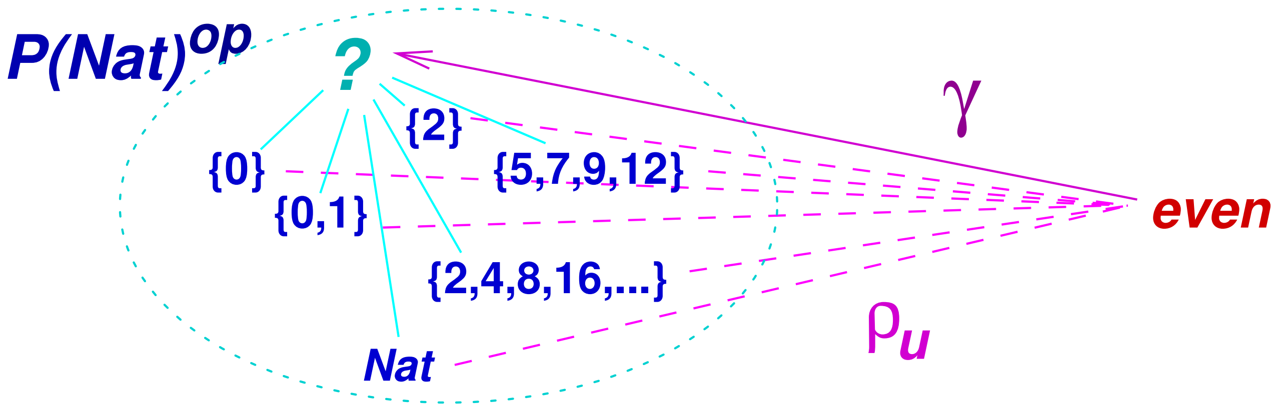


then the under-approximating $\mathit{even} \in \mathit{Parity}^{op}$ *should assert* *"*$\exists \mathit{even}$*"*— there exists an even number in the program's outputs:



For the Collatz example, this lets us define $f^{\flat}(\mathrm{odd}) = \mathit{even}$ — for every odd-valued argument, there exists an even-valued answer.

But we cannot define $\gamma : \mathrm{Parity}^{op} \rightarrow \mathcal{P}(\mathrm{Nat})^{op}$ in the usual way:



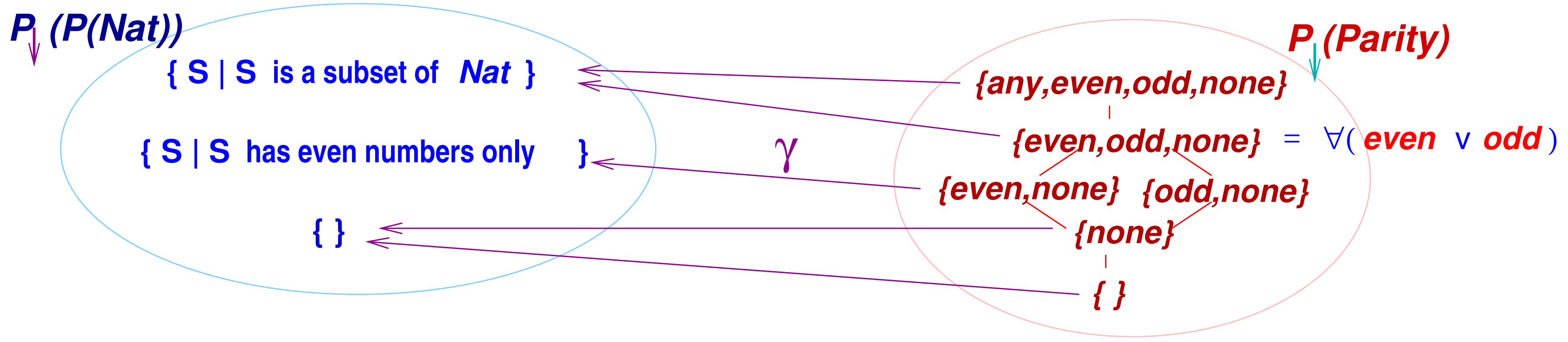
There is no best, minimal set that contains an even number.

Indeed, even's concretization is not a single set — it is a *set of sets*:

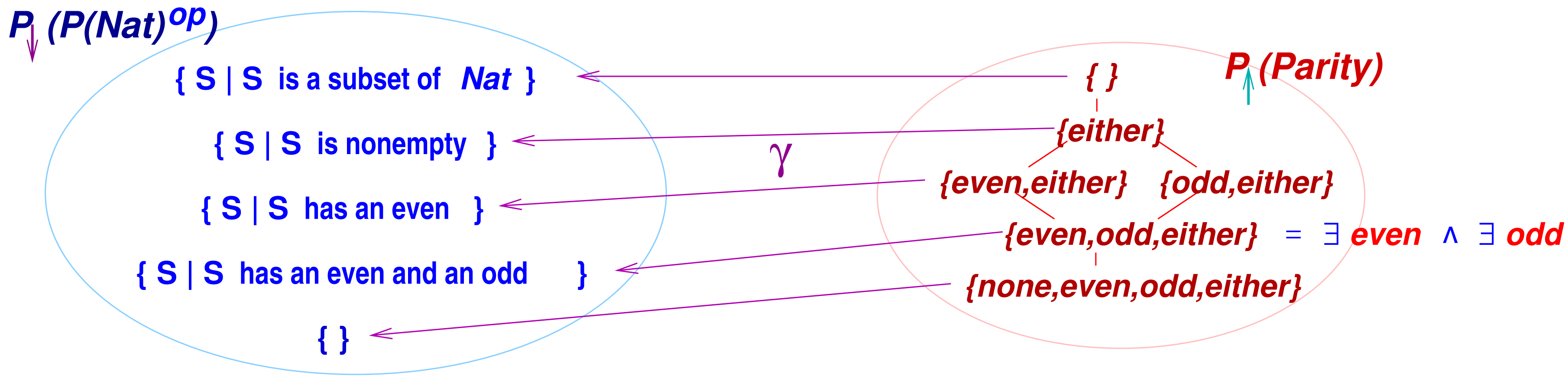
$$\gamma(\mathit{even}) = \{S \in \mathcal{P}(\mathrm{Nat})^{op} \mid S \ \rho_u \ \mathit{even}\}$$

This suggests that we work with *power-domains* in both the concrete and abstract domains.

Universal (over-approximating) interpretation: $\{\mathrm{even}, \mathrm{odd}\}$ asserts $\forall\{\mathrm{even}, \mathrm{odd}\} \equiv \forall(\mathrm{even} \vee \mathrm{odd})$ — all outputs are even- or odd-valued: Use a lower power-domain (lower sets) for the abstract domain.



Existential (under-approximating) interpretation: $\{\mathrm{even}, \mathrm{odd}\}$ asserts $\exists\{\mathrm{even}, \mathrm{odd}\} \equiv \exists\mathrm{even} \wedge \exists\mathrm{odd}$ — there exists an even-valued and an odd-valued output: Use an upper power-domain (upper sets).



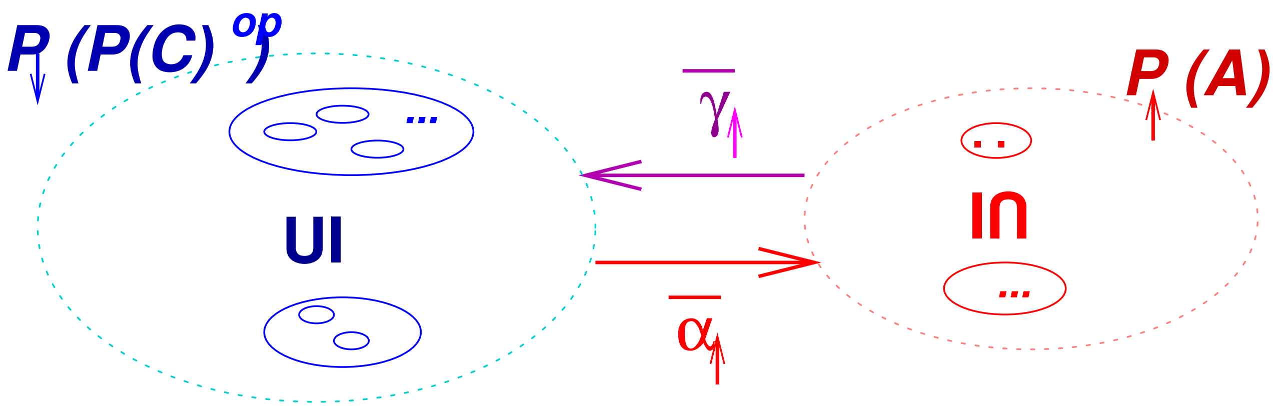
Existential approximation uses the
Smyth-powerdomain ordering

For concrete values, $S \subseteq C$, and abstract values, $T \subseteq A$,

$$S\ \rho_\uparrow\ T \text{ iff for all } a \in T\text{, there exists } c \in S \text{ such that } c \in \gamma(a)$$

Every $a \in T$ is a witness to some $c \in S$. (Smyth-powerdomain ordering)

$\overline{\gamma}_\uparrow(T) = \{S \mid S\ \rho_\uparrow\ T\}$ concretizes T to all sets that T "witnesses" — *It is an overapproximation of an underapproximation:*



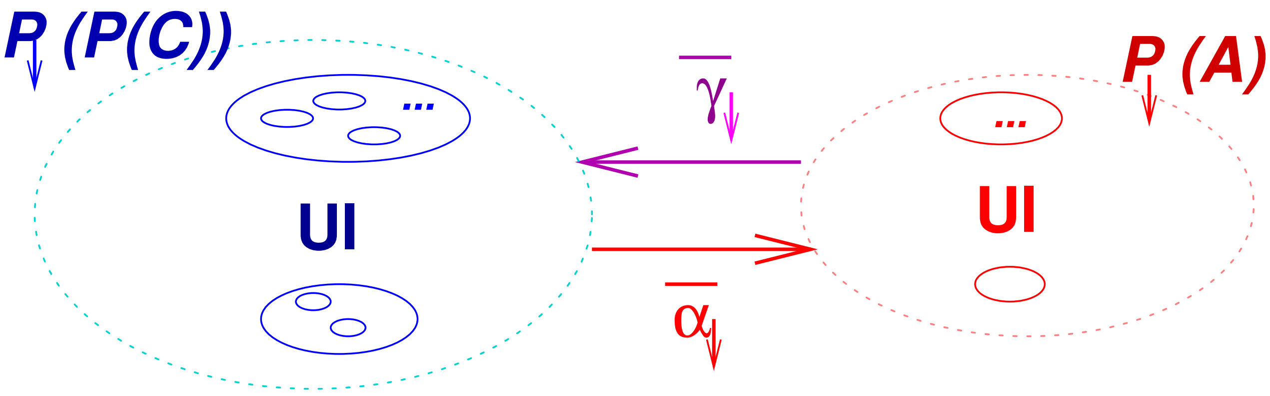
$$\begin{aligned}\overline{\alpha}_\uparrow(\overline{S}) &= \cup\{T \mid \text{ for all } S \in \overline{S}, S\ \rho_\uparrow\ T\}\\ &= \{a \mid \text{ for all } S \in \overline{S}, \text{ exists } c \in S, c \in \gamma(a)\}\end{aligned}$$

Universal approximation uses the lower-powerdomain ordering

$S \; \rho_{\downarrow} \; T$ iff for all $c \in C$, there exists $a \in A$ such that $c \in \gamma(a)$

Every $c \in S$ is modelled by some $a \in T$ — (lower ("Hoare") powerdomain ordering)

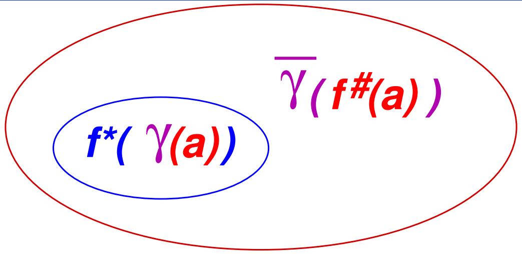
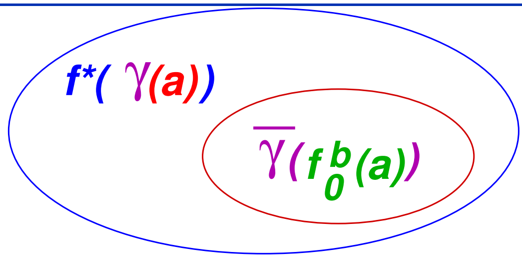
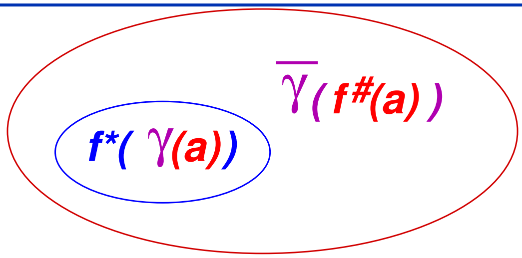
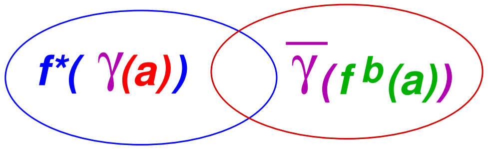
$\overline{\gamma}_{\downarrow}(T) = \{S \mid S \; \rho_{\downarrow} \; T\}$ concretizes T to all the sets covered by T — *It is an overapproximation of an overapproximation:*



$\overline{\alpha}_{\downarrow}(\overline{S}) = \cap\{T \mid$ for all $S \in \overline{S}, S \; \rho_{\downarrow} \; T\}$

Summary: *Forms of approximation*

For $f : C \to \mathcal{P}(C)$ and its lift, $f^* : \mathcal{P}(C) \to \mathcal{P}(C)$,
for $f^\sharp : A \to \mathcal{P}_\downarrow(A)$, $f_0^\flat : A \to \mathcal{P}_\downarrow(A)$, and $f^\flat : A \to \mathcal{P}_\uparrow(A)$,

	overapproximation	**underapproximation**
set inclusion	 $f^*(\gamma(a)) \subseteq \overline{\gamma}(f^\sharp(a))$	 $f^*(\gamma(a)) \supseteq \overline{\gamma}(f_0^\flat(a))$
quantification	 $f^*(\gamma(a)) \models \forall(f^\sharp(a))$	 $f^*(\gamma(a)) \models \exists(f^\flat(a))$

- Read $\forall(f^\sharp(a)) \equiv \forall\{a_0, a_1, \cdots, a_i, \cdots\} \equiv \forall(a_0 \vee a_1 \vee \cdots \vee a_i \vee \cdots)$.
- Read $\exists(f^\flat(a)) \equiv \exists\{a_0, a_1, \cdots, a_i, \cdots\} \equiv \exists a_0 \wedge \exists a_1 \wedge \cdots \wedge \exists a_i \wedge \cdots$.

Summary: *Approximating the logic*

$$\phi ::= p \mid \cdots \mid \Box\phi \mid \Diamond\phi$$

$$[\![p]\!]^A = \overline{\alpha}_u [\![\phi]\!]$$

$$[\![\Box\phi]\!]^A = \tilde{\mathrm{pre}}_{f^\sharp} [\![\phi]\!]^A$$

$$[\![\Diamond\phi]\!]^A = \mathrm{pre}_{f^\flat} [\![\phi]\!]^A$$

where $f^\sharp : A \to \mathcal{P}_\downarrow(A)$ and $f^\flat : A \to \mathcal{P}_\uparrow(A)$.

To overapproximate $f : C \to \mathcal{P}(C)$**, use** $f^\sharp_{best} : A \to \mathcal{P}_\downarrow(A)$**,**

$$f^\sharp_{best} = \overline{\alpha}_o \circ f^* \circ \gamma = \overline{\alpha}_\downarrow \circ (\{\!|\cdot|\!\} \circ f)^* \circ \gamma$$

In this thesis, Dams defined $f^\sharp_{best}(a) = \downarrow\{\alpha\{c'\} \mid c' \in f(c), c \in \gamma(a)\}$.

To underapproximate $f : C \to \mathcal{P}(C)$**, use** $f^\flat_{best} : A \to \mathcal{P}_\uparrow(A)$**,**

$$f^\flat_{best} = \overline{\alpha}_\uparrow \circ (\{\!|\cdot|\!\} \circ f)^* \circ \gamma$$

In his thesis, Dams defined $f^\flat_{best}(a) = \{a' \mid \gamma(a') \cap f(c) \neq \emptyset,$ for all $c \in \gamma(a)\}$.

Dams, Gerth, and Grumberg proved that these definitions soundly validate the most logical properties of f.

We can also validate LTL: $\psi ::= p \mid F\psi \mid G\psi \mid \cdots$

$$[\![p]\!] = \{\pi \in C^\infty \mid \pi_0 \in \gamma(p)\}$$

$$[\![F\psi]\!] = \{\pi \in C^\infty \mid \text{ exists } i \geq 0, \pi_i \in [\![\psi]\!]\}$$

$$[\![G\psi]\!] = \{\pi \in C^\infty \mid \text{ for all } i \geq 0, \pi_i \in [\![\psi]\!]\}$$

Use $f^\sharp$ to generate paths of form, $\pi^\sharp = (a_i)_{i \geq 0}$, such that for all $i \geq 0$, ${\pi^\sharp}_{i+1} \in f^\sharp({\pi^\sharp}_i)$. **Validate**

$a_0 \models \forall\psi$ **"for all paths, $\pi^\sharp$ starting with a_0, $\pi^\sharp \models \psi$"**

to prove $c_0 \models \forall\psi$, for all $c \in \gamma(a_0)$, since $f \lhd f^\sharp$.

Use $f^\flat$ to generate paths $\pi^\flat = (a_i)_{i \geq 0}$, such that for all $i \geq 0$, ${\pi^\flat}_{i+1} \in f^\flat({\pi^\flat}_i)$. **Validate**

$a_0 \models \exists\psi$ **"there exists a path, $\pi^\flat$, starting with a_0, such that $\pi^\flat \models \psi$"**

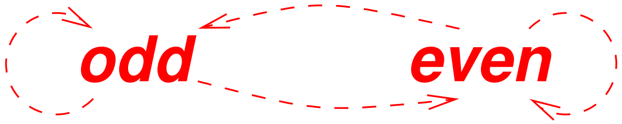
to prove $c_0 \models \exists\psi$, for all $c \in \gamma(a_0)$, since $f^\flat \lhd f$.

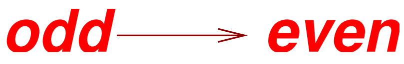
Mixed modalities and mixed models

To prove that transitions from even-valued states never "go far,"

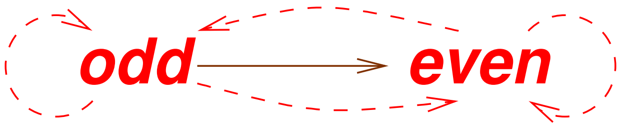
$$even \models^{A} \Box(even \vee \Diamond even)$$

we need more than just a sound overapproximation alone or a sound underapproximation alone:





Working with the two together — as a *mixed model* — we can validate the claim:



Definition: For Kripke structure, (C, f, γ), a *mixed model* is $(A, f^{\flat}, f^{\sharp})$ such that $f^{\flat} \lhd f$ and $f \lhd f^{\sharp}$.

Other extensions

The validation logic can contain repetition/recursion:

$$\phi ::= p \mid \Box\phi \mid \Diamond\phi \mid \cdots \mid \mu Z.\phi \mid \nu Z.\phi \mid Z$$

Sound checking is preserved on the mixed models.

We can define standard modalities, e.g.,

$$AG_{fin}\phi \equiv \nu Z.\phi \wedge \Box Z$$

$$AG\phi \equiv \nu Z.\phi \wedge \Diamond true \wedge \Box Z$$

and use them:

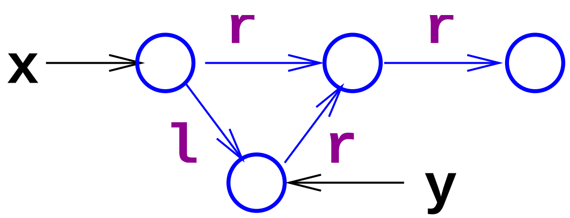
$$even \models^{A} AG(even \vee \Diamond even)$$

Finally, we can have more than one state-transition function, g, and the corresponding logical modalities, $[g]\phi$ and $\langle g\rangle\phi$, as in Hennessy-Milner logic and description logic.

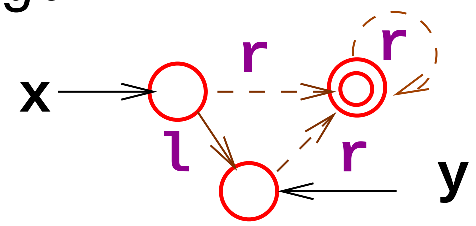
Underapproximating data structure

TVLA models of object networks [Sagiv,Reps,Wilhelm]

A collection of data objects,



can be modelled by partitioning the objects based on their properties and abstracting their linkage:



Say we have $\gamma : \mathrm{Prop} \to \mathcal{P}(\mathrm{Object})$, e.g., $\mathrm{Prop} = \{\mathrm{x\_pointsTo}, \mathrm{y\_pointsTo}\}$.

Partition Object by $c \equiv_\gamma c'$ iff (for all $p \in \mathrm{Prop}$, $c \in \gamma(p)$ iff $c' \in \gamma(p)$).

All partitions are nonempty **— distinguish between *singleton* partitions, ○, and *multiple-object* partitions, ◎.**

Each fieldname defines a "transition function"

Abstract a field function, $f : C \to C$, **as** $f^\sharp : A \times A \to \{0, 1, 1/2\}$:

$f^\sharp(a, b) = 1$: for all $c \in \gamma(a)$, for all $c' \in \gamma(b)$, $f(c) = c'$ $a \xrightarrow{f} b$

$f^\sharp(a, b) = 1/2$: there exist $c \in \gamma(a)$, $c' \in \gamma(b)$, such that $f(c) = c'$ $a \overset{f}{\dashrightarrow} b$

$f^\sharp(a, b) = 0$: otherwise (there are no $c \in \gamma(a)$, $c' \in \gamma(b)$, such that $f(c) = c'$) $a \quad b$

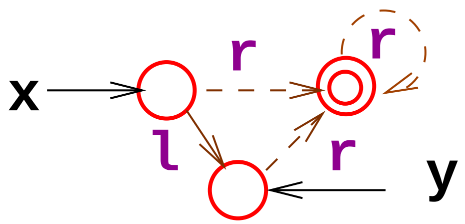
- TVLA is an *over*approximating model, where some nodes and links are *exact*.
- We might rework TVLA's approximations as $f^\flat : A \to \mathcal{P}_\uparrow(A)$ (for 1 values) and $f^\sharp : A \to \mathcal{P}_\downarrow(A)$ (for $1/2$ *and* 1 values).
- $f^\sharp(a, b) = 1$ is a $\forall\forall$-abstraction of f (the usual $f^\flat$ is a $\forall\exists$-abstraction), but $\forall\forall$ *coincides* with $\forall\exists$ in TVLA models, where concrete graphs are deterministic.

Logical properties of the shape graph

TVLA uses first-order predicate logic plus transitive closure:

$$\phi ::= p_k(e_i)_{i<k} \mid \cdots \mid \forall x.\phi \mid \exists x.\phi \mid p^+(e_1, e_2)$$

Example: for



$$\exists i.\ \mathtt{x\_pointsTo}(i) \wedge \exists j.\ \mathtt{l}(i,j) \wedge \neg(\exists k.\ \mathtt{r}^+(j,k) \wedge \mathtt{x\_pointsTo}(k))$$

holds true: *Starting from* `x`*'s* `l`*-field, repeated transitions of the* `r`*-field never lead back to* `x`*'s object.*

We could also use a Box-Diamond logic, like that used to state properties of control diagrams, to state properties of the shape graph:

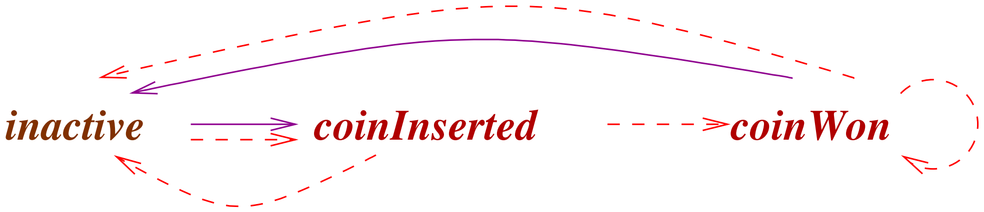
$$\mathtt{x} \models \langle \mathtt{l} \rangle AG_r \neg \mathtt{x\_pointsTo}, \quad \text{where } AG_r\phi = \nu Z.\phi \wedge [\mathtt{r}]Z$$

Underapproximation in specification

Modal transition systems *[Larsen,Thomson]*

A reactive system can be specified with an under-over-approximation.

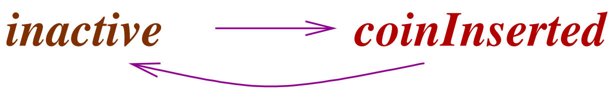
A slot-machine specification:



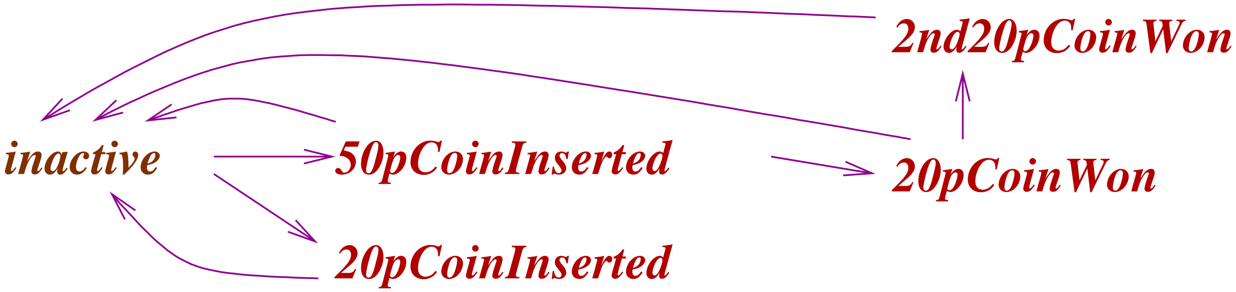
Must ("tight")-transitions indicate behaviors that the completed machine must possess; *may ("loose")*-transitions indicate the domain of acceptable (but not required) behaviors.

Two possible implementations:

(i) A machine that keeps all coins:



(ii) A machine that might pay one or two 20p coins for a 50p coin:



Definition: A modal transition system (MTS) is a mixed-transition system, (C, f_{must}, f_{may}), such that $f_{must}(c) \subseteq f_{may}(c)$, for all $c \in C$.

Every must-transition is also a may-transition (“what must be implemented, surely may be implemented”).

An MTS can be stepwise refined into an implementation — an STS (this is an MTS where $f_{must} = f_{may}$) — via *refinement*:

Definition: For modelling relation, $\gamma : C' \to \mathcal{P}(C)$, $(C, f_{must}, f_{may}) \lhd (C', f'_{must}, f'_{may})$ iff $f_{may} \lhd f'_{may}$ and $f'_{must} \lhd f_{must}$.

May-transitions are preserved or deleted, and must-transitions are preserved/increased.

We can use the logics stated earlier to write properties of the MTS.

Description logic [Baader, et al.]

Description logic is used to specify knowledge bases:

$$\phi \ ::= \ p \mid \phi_1 \sqcup \phi_2 \mid \phi_1 \sqcap \phi_2 \mid \forall R.\phi \mid \exists R.\phi \mid \ \geq n.R \mid \ \leq n.R$$

Read $\forall R.\phi$ as $[R]\phi$ and $\exists R.\phi$ as $\langle R \rangle \phi$. Read $c \models \ \geq n.R$ iff $|\{c' \mid R(c, c')\}| \geq n$.

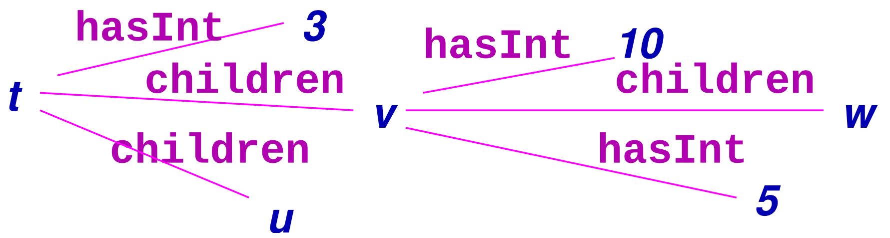
Example: $\text{Tbox}:$ $\quad \mathrm{IntTree} \equiv \ \geq 0.\mathrm{hasInt} \ \sqcap \ \forall \mathrm{children}.\mathrm{IntTree}$

$$\mathrm{BinTree} \equiv \ \mathrm{IntTree} \sqcap \ \leq 2.\mathrm{children}$$

$\text{Abox}:$ $\quad \mathrm{hasInt}(t, 3) \ \ \mathrm{children}(t, u) \ \ \mathrm{children}(t, v)$

$$\mathrm{hasInt}(v, 10) \ \ \mathrm{hasInt}(v, 5) \ \ \mathrm{children}(v, w)$$

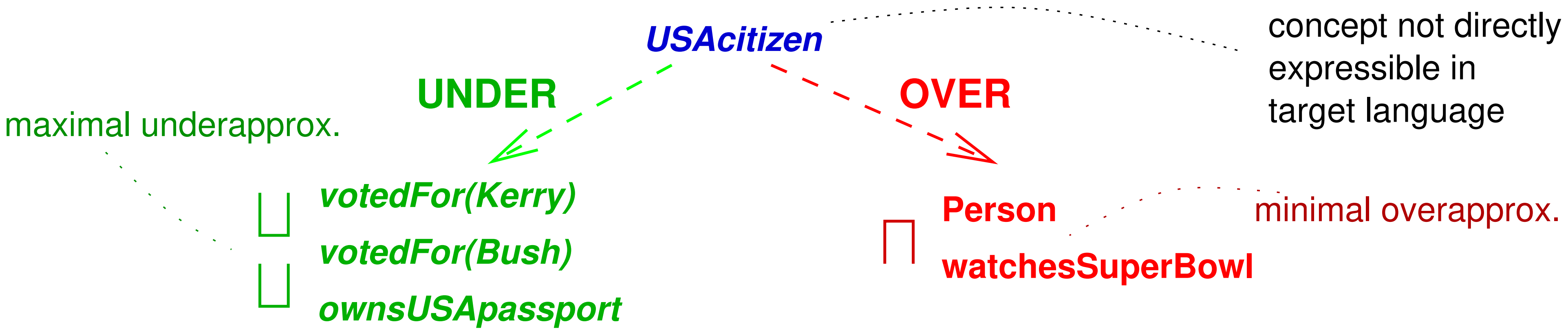
Description logic is a “super Prolog” whose inference engine is a model checker!



Is the example an *over-* or an *under-* specification ?

In his thesis, Ciocoiu uses description logic as a meta-language for *inexact language translation*:

Both source and target languages are given description-logic semantics; translation of a source sentence into the target language produces an *overapproximation* translation and an *underapproximation* translation:

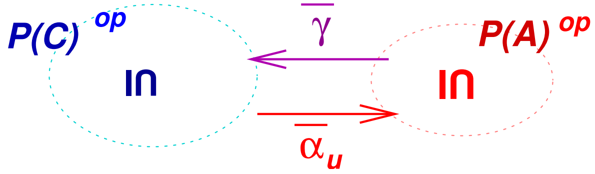


Taken together, the two inexact translations describe the source concept in the target language.

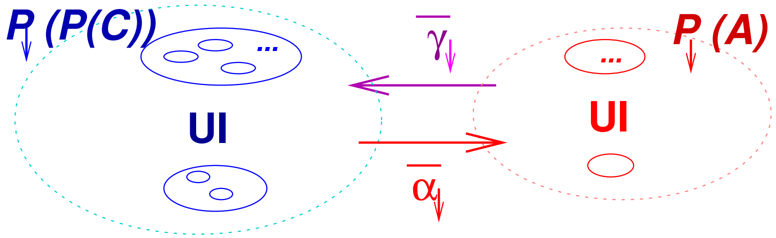
Conclusion

Explaining the slogan: "Overapproximate the computation and underapproximate the logic"

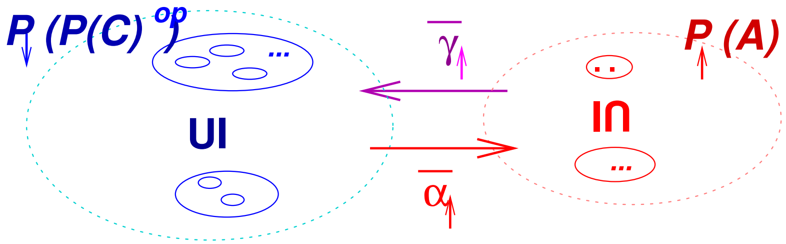
- the *logic* is underapproximated with $\mathcal{P}_\downarrow(A)^{\mathrm{op}}$:



- a *computation* is approximated for universal properties with $\mathcal{P}_\downarrow(A)$:



- a *computation* is approximated for existential properties with $\mathcal{P}_\uparrow(A)$:



- *Both* of the previous approximations use concrete domains of form, $\mathcal{P}_\downarrow(\mathcal{P}(\cdot))$, making them *over* approximations of the concrete system.

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