Underapproximating Predicate Transformers

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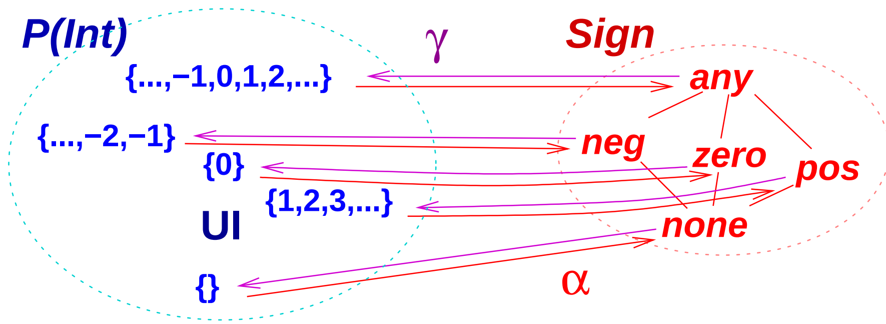
Background

Given 

and $\mathrm{succ} : \mathrm{Int} \rightarrow \mathcal{P}(\mathrm{Int})$**,** $\mathrm{succ}(n) = \{n+1\}$**, let's validate** $\mathrm{succ}(0) \subseteq \{1,2,3,\cdots\}$ **within** Sign**:**

1. We approximate succ by $\mathrm{succ}^{\sharp}_{\mathrm{best}} = \alpha \circ \mathrm{succ}^{*} \circ \gamma$, and we approximate 0 by $\alpha\{0\} = \mathit{zero}$.
2. We approximate $\{1,2,3,\cdots\}$ by $\alpha\{1,2,3,\cdots\} = \mathit{pos}$ $(*)$
3. We check that $\mathrm{succ}^{\sharp}_{\mathrm{best}}(\mathit{zero}) \sqsubseteq_{\mathit{Sign}} \mathit{pos}$. (It does.)

$(*)$ Step 2 is sound only if the property, $S \subseteq \mathrm{Int}$, is *exact*: $S = \gamma(\alpha(S))$.

For example, $\alpha\{-2,0\} = \mathit{any}$, but $\gamma(\mathit{any}) \neq \{-2,0\}$. Therefore,

$\mathrm{succ}^{\sharp}_{\mathrm{best}}(\mathit{zero}) \sqsubseteq_{\mathit{Sign}} \mathit{any}$ *does not imply* $\mathrm{succ}(0) \in \{-2,0\}$.

A logic whose assertions are exact

For Galois connection, $(\mathcal{P}(C), \subseteq)\langle\alpha, \gamma\rangle(A, \sqsubseteq)$, define this logic:

$$\phi ::= a \mid \phi_1 \sqcap \phi_2 \text{ , where } a \in A$$

Each ϕ is interpreted as $[\![\phi]\!] = \gamma(\phi)$, so that $a \sqsubseteq \phi$ implies $\gamma(a) \subseteq [\![\phi]\!]$. That is, the sets, $\gamma(\phi)$, are exact.

This makes $f^\sharp_{best} : A \to A$ the strongest postcondition transformer for $f : C \to \mathcal{P}(C)$ in "logic" A:

$$f^*[\![\phi]\!] \subseteq [\![\phi']\!] \text{ iff } f^\sharp_{best}(\phi) \sqsubseteq \phi'$$

That is, $\{\phi\}\, f\, \{f^\sharp_{best}(\phi)\}$ is a sound and complete Hoare-triple for f [Cousot78] .

But such logics are rare

$$\phi ::= \mathrm{neg} \mid \mathrm{zero} \mid \mathrm{pos} \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2$$

$$[\![\mathrm{neg}]\!] = \gamma(\mathrm{neg}) \qquad [\![\mathrm{zero}]\!] = \gamma(\mathrm{zero}) \qquad [\![\mathrm{pos}]\!] = \gamma(\mathrm{pos})$$

$$[\![\phi_1 \wedge \phi_2]\!] = [\![\phi_1]\!] \cap [\![\phi_2]\!]$$

$$[\![\phi_1 \vee \phi_2]\!] = [\![\phi_1]\!] \cup [\![\phi_2]\!]$$

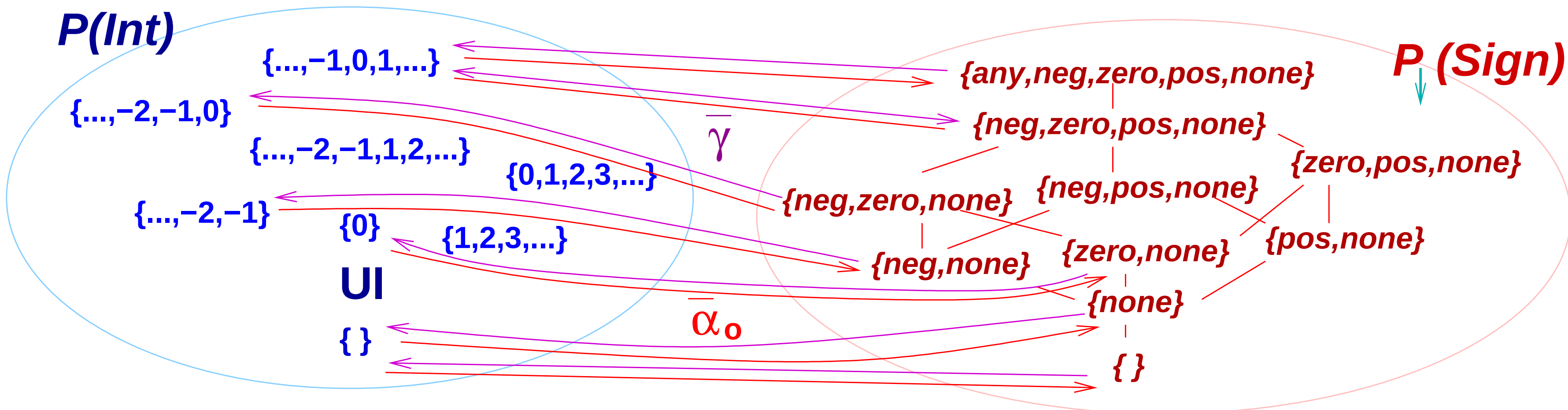
For $(\mathcal{P}(\mathrm{Int}), \subseteq)\langle\alpha, \gamma\rangle(\mathit{Sign}, \sqsubseteq)$, disjunction is not exact, e.g.,

$[\![\mathrm{neg} \vee \mathrm{zero}]\!] = \{\cdots, -2, -1, 0\} \subset \gamma(\alpha\{\cdots, -2, -1, 0\}) = \gamma(\mathrm{any}) = \mathrm{Int}$.

Therefore, we dare not approximate the assertion, $[\![\mathrm{neg} \vee \mathrm{zero}]\!]$, by $\alpha[\![\mathrm{neg} \vee \mathrm{zero}]\!] = \mathrm{any} \in \mathit{Sign}$, because it is an *overapproximation*

Sometimes, we can complete the abstract domain

We construct the *disjunctive completion* [Cousots79,Giacobazzi00] :



$$(\mathcal{P}(\mathrm{int}), \subseteq)\langle\overline{\alpha_o}, \overline{\gamma}\rangle(\mathcal{P}_{\downarrow}(\mathrm{Sign}), \subseteq)$$

$$\overline{\gamma}(T) = \cup_{a \in T}\gamma(a) \qquad \overline{\alpha_o}(S) = \downarrow\{\alpha\{c\} \mid c \in S\}$$

Downclosed sets are needed for monotonicity of key functions on the sets.

We interpret properties, $S \subseteq \mathrm{Int}$, as $\overline{\alpha_o}(S) \in \mathcal{P}_{\downarrow}(A)$ and functions, $f : \mathrm{Int} \to \mathcal{P}(\mathrm{Int})$, as $f^{\sharp}_{best} = (\overline{\alpha_o} \circ f^* \circ \gamma) \in A \to \mathcal{P}_{\downarrow}(A)$.

Now, all assertions are exact: $\phi ::= neg \mid zero \mid pos \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2$

E.g., $\overline{\alpha_o}[\![neg \vee zero]\!] = \downarrow\{neg, zero\} = \{neg, zero, none\}$.

Sometimes, the completion is too expensive

For $succ : \mathrm{Int} \rightarrow \mathcal{P}(\mathrm{Int})$, define the precondition assertion $[succ]\phi$ as

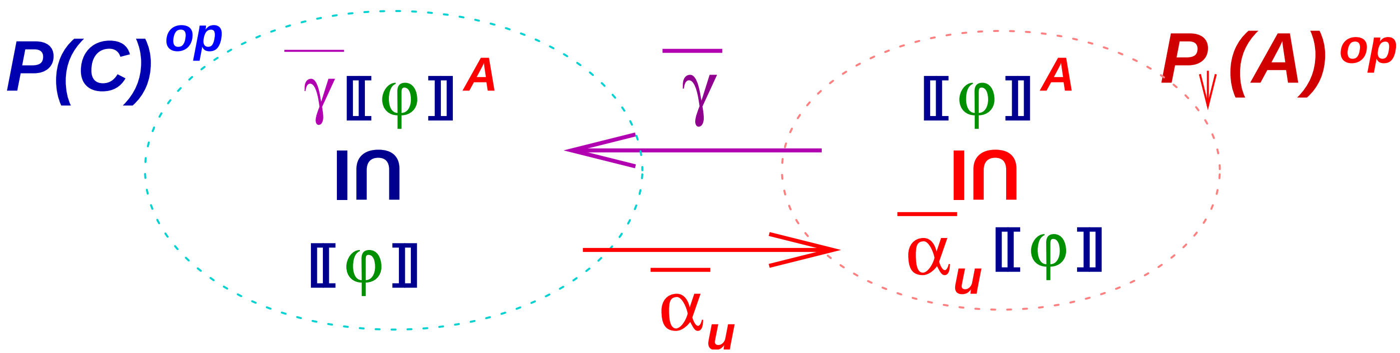
$$[\![[succ]\phi]\!] = \{n \mid succ(n) \subseteq [\![\phi]\!]\}$$

Because $zero \in Sign$, the completion of $Sign$ with respect to $[succ]zero$ *adds each and every negative integer,* $-1, -2, \cdots$ to the abstract domain. (because we must make $[succ]zero$ exact, and then make $[succ][succ]zero$ exact, etc.)

As an alternative, we *underapproximate* nonexact assertions.

How do we underapproximate $[\![\cdot]\!] : \mathcal{L} \to \mathcal{P}(C)$?

Lift the original Galois connection, $\mathcal{P}(C)\langle\alpha,\gamma\rangle A$, to [Cousots00] :



The best abstraction of $[\![\phi]\!]$ is merely

$$[\![\phi]\!]^A = \overline{\alpha_u}[\![\phi]\!],$$

because $[\![\phi]\!] \supseteq \overline{\gamma}(\overline{\alpha_u}[\![\phi]\!])$.

Example: for $[\![\phi_1 \vee \phi_2]\!] = [\![\phi_1]\!] \cup [\![\phi_2]\!]$, we have

$$\begin{aligned}[\![\phi_1 \vee \phi_2]\!]^{Sign} &= \overline{\alpha_u}[\![\phi_1 \vee \phi_2]\!] \\ &= \{a \in Sign \mid \gamma(a) \subseteq [\![\phi_1]\!] \cup [\![\phi_2]\!]\}.\end{aligned}$$

But this is not defined inductively on $[\![\phi]\!]^A$.

Define the abstract logic inductively

For assertion, $\mathrm{op}_g(\phi_i)_{i<k}$, interpreted as

$$[\![\mathrm{op}_g(\phi_i)_{i<k}]\!] = g([\![\phi_i]\!])_{i<k}, \quad \text{where } g : \mathcal{P}(C)^k \to \mathcal{P}(C),$$

interpret it abstractly as

$$[\![\mathrm{op}_g(\phi_i)_{i<k}]\!]^{A}_{ind} = (\overline{\alpha_u} \circ g \circ \overline{\gamma}^k)([\![\phi_i]\!]^{A}_{ind})_{i<k}$$

We have $\overline{\alpha_u}[\![\phi]\!] \supseteq [\![\phi]\!]^{A}_{ind}$, and when ϕ is exact, $\supseteq$ becomes $=$.

Example: $[\![\phi_1 \vee \phi_2]\!] = [\![\phi_1]\!] \cup [\![\phi_2]\!]$ abstracts to

$$[\![\phi_1 \vee \phi_2]\!]^{Sign}_{ind} = (\overline{\alpha_u} \circ \cup \circ \overline{\gamma}^2)([\![\phi_1]\!]^{Sign}_{ind}, [\![\phi_2]\!]^{Sign}_{ind}).$$

Can we eliminate concrete $\cup$ *from* $[\![\phi_1 \vee \phi_2]\!]^{Sign}_{ind}$*?* Try this:

$$[\![\phi_1 \vee \phi_2]\!]^{Sign} = [\![\phi_1]\!]^{Sign} \cup_{\mathcal{P}_\downarrow(Sign)} [\![\phi_2]\!]^{Sign}.$$

But $\cup_{\mathcal{P}_\downarrow(Sign)} \neq (\overline{\alpha_u} \circ \cup \circ \overline{\gamma}^2)$ — we lose precision:

$any \in [\![neg \vee zero \vee pos]\!]^{Sign}_{ind}$, yet $any \notin [\![neg]\!]^{Sign} \cup [\![zero]\!]^{Sign} \cup [\![pos]\!]^{Sign}$

A more difficult example — *preconditions*: for $f : C \to \mathcal{P}(C)$,

$$[\![[f]\phi]\!] = \widetilde{pre}_f [\![\phi]\!],$$

$$\text{where } \widetilde{pre}_f(S) = \{c \mid f(c) \subseteq S\},$$

we have

$$\begin{aligned} [\![[f]\phi]\!]^A_{ind} &= (\overline{\alpha_u} \circ \widetilde{pre}_f \circ \overline{\gamma}) [\![\phi]\!]^A_{ind} \\ &= \{a \mid f^*[\gamma(a)] \subseteq \overline{\gamma} [\![\phi]\!]^A_{ind}\}. \end{aligned}$$

Is this finitely computable? Can it be expressed compositionally as

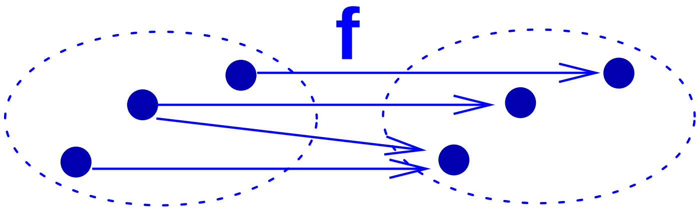
$$[\![[f]\phi]\!]^A = \widetilde{pre}_{f^\sharp} [\![\phi]\!]^A$$

for some $f^\sharp : A \to \mathcal{P}(A)$? Do we lose precision?

This is the topic of the paper in the SAS'06 proceedings.

Defining sound underapproximations of predicate transformers used in dynamic and temporal logic

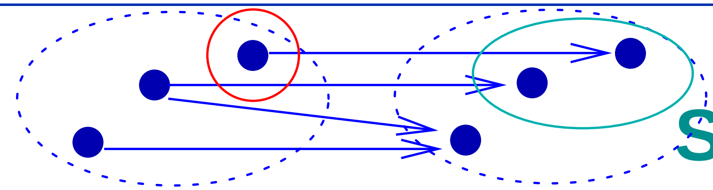
For nondeterministic state-transition function, $f : C \rightarrow \mathcal{P}(C)$,



, and property $S \subseteq C$, we have

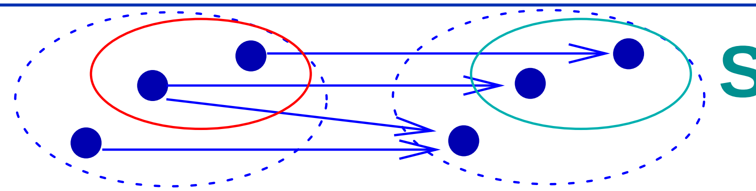
$\widetilde{\mathrm{pre}}_f(S) = \{c \mid f(c) \subseteq S\}$

"forall precondition": transit only into S



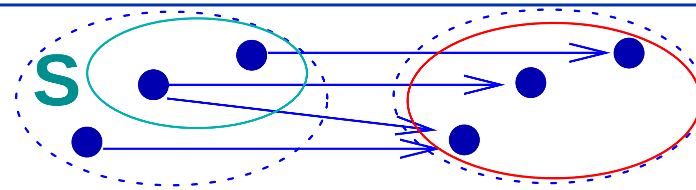
$\mathrm{pre}_f(S) = \{c \mid f(c) \cap S \neq \emptyset\}$

"exists precondition": transit to S



$\mathrm{post}_f(S) = f^*(S)$

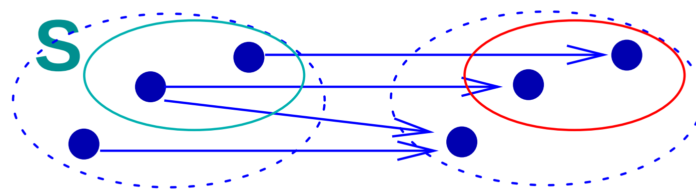
"exists postcondition": are reached by S



$\widetilde{\mathrm{post}}_f(S)$

$= \{d \mid \forall c \in C, d \in f(c) \Rightarrow c \in S\}$

"forall postcondition": are reached only by S



$\widetilde{\mathrm{pre}}, \mathrm{post}$ are used for validation; $\mathrm{pre}, \widetilde{\mathrm{post}}$ are used for code improvement

The transformers interpret this logic

$$\phi ::= a \mid \cdots \mid [f]\phi \mid \langle f\rangle\phi \mid \phi\overline{[f]} \mid \phi\overline{\langle f\rangle}$$

as follows:

$$[\![[f]\phi]\!] = \widetilde{pre}_f[\![\phi]\!] \qquad [\![\phi\overline{[f]}]\!] = \widetilde{post}_f[\![\phi]\!]$$

$$[\![\langle f\rangle\phi]\!] = pre_f[\![\phi]\!] \qquad [\![\phi\overline{\langle f\rangle}]\!] = post_f[\![\phi]\!]$$

Although these are "single-step" assertions, we use recursion to define interesting properties, like those in *CTL*:

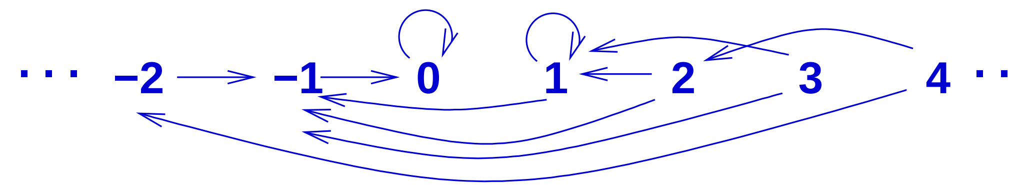
$AG_f\phi \equiv \nu Z.\phi \wedge [f]Z$ for all f-transition sequences, ϕ holds

$EF_f\phi \equiv \mu Z.\phi \vee \langle f\rangle Z$ there exists an f-transition sequence leading to ϕ

$\phi\overline{EF}_f \equiv \mu Z.\phi \vee Z\overline{\langle f\rangle}$ there exists an f-transition sequence from ϕ to here

Example: ***Transition function*** $\mathrm{h} : \mathrm{Int} \to \mathcal{P}(\mathrm{int})$

```
let h(n) = if neg(n) :
               n:= n+1
           else truncate(sqrt(n))
in loopforever h
```



Some properties of h:

$[\![[\mathrm{h}]\mathrm{neg}]\!] = \widetilde{\mathrm{pre}}_{\mathrm{h}}\{\cdots, -2, -1\} = \{\cdots, -3, -2\}$ transit only into negatives

$[\![\langle \mathrm{h} \rangle \mathrm{neg}]\!] = \mathrm{pre}_{\mathrm{h}}\{\cdots, -2, -1\} = \{\cdots, -3, -2, 1, 2, 3, \cdots\}$ transit to a negative

$[\![\mathrm{neg}\overline{\langle \mathrm{h} \rangle}]\!] = \mathrm{post}_{\mathrm{h}}\{\cdots, -2, -1\} = \{\cdots, -2, -1, 0\}$ are reached by negatives

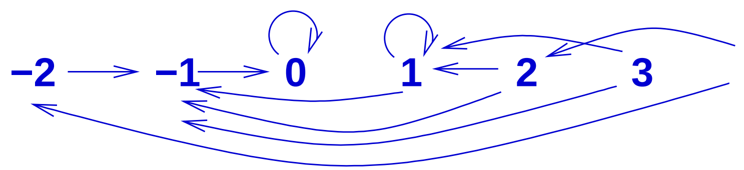
$[\![\mathrm{neg}\overline{[\mathrm{h}]}]\!] = \widetilde{\mathrm{post}}_{\mathrm{h}}\{\cdots, -2, -1\} = \{\}$ are reached only by negatives

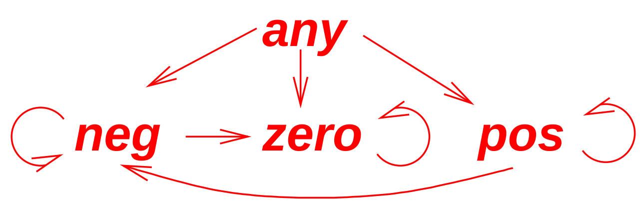
Underapproximating $\widetilde{\mathsf{pre}}_f(S) = \{c \mid f(c) \subseteq S\}$

Theorem: $(\overline{\alpha_u} \circ \widetilde{\mathsf{pre}}_f \circ \overline{\gamma}) = \widetilde{\mathsf{pre}}_{f^\sharp_{best}}$, where $f^\sharp_{best} = \overline{\alpha_o} \circ f^* \circ \gamma$.

Intuition: $f^\sharp$'s preimage overapproxes f's, and $[\![\phi]\!]^A$ underapproxes $[\![\phi]\!]$.

$$[\![[f]\phi]\!]^A_{ind} = (\overline{\alpha_u} \circ \widetilde{\mathsf{pre}}_f \circ \overline{\gamma})[\![\phi]\!]^A_{ind} = \widetilde{\mathsf{pre}}_{f^\sharp_{best}}[\![\phi]\!]^A_{ind}$$

Example: $h = \cdots$  $\cdots$

$h^\sharp_{best} =$ 

What must transit to zero ? $[\![[h]zero]\!] = \{-1, 0\}$

The approximation is $[\![[h]zero]\!]^{Sign}_{ind} = \downarrow\{zero\}$

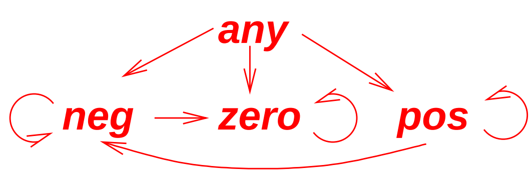
The abstraction of $\widetilde{\mathsf{pre}}_h$ is the best we can do, but it loses precision.

Underapproximating $\mathrm{pre}_f(S) = \{c \mid f(c) \cap S \neq \emptyset\}$

$$[\![\langle f\rangle\phi]\!]^{A}_{ind} = (\overline{\alpha_u} \circ \mathrm{pre}_f \circ \overline{\gamma})[\![\phi]\!]^{A}_{ind}$$

But, for $f^\sharp : A \to \mathcal{P}_\downarrow(A)$, $\mathrm{pre}_{f^\sharp}$ can be *unsound* ! Intuition: $h^\sharp$ overestimates h's preimage, so there can be “false transitions.”

Example: $[\![\langle h\rangle neg]\!] = \mathrm{pre}_h[\![neg]\!] = \{\cdots, -3, -2, 1, 2, 3\}$ transit to negatives.

But $h^\sharp_{best} =$ 

, and

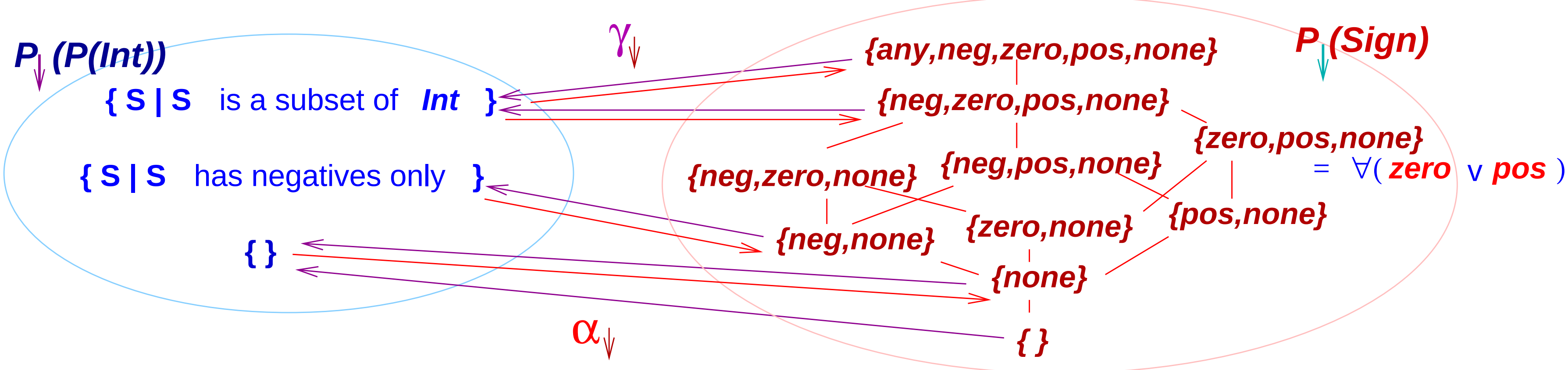
$\mathrm{pre}_{h^\sharp_{best}}[\![neg]\!] = \{neg, pos, any\}$ and $\overline{\gamma}\{neg, pos, any\} = \mathrm{Int}$!

Computational approximation with downclosed sets is incorrect *for* pre*:*

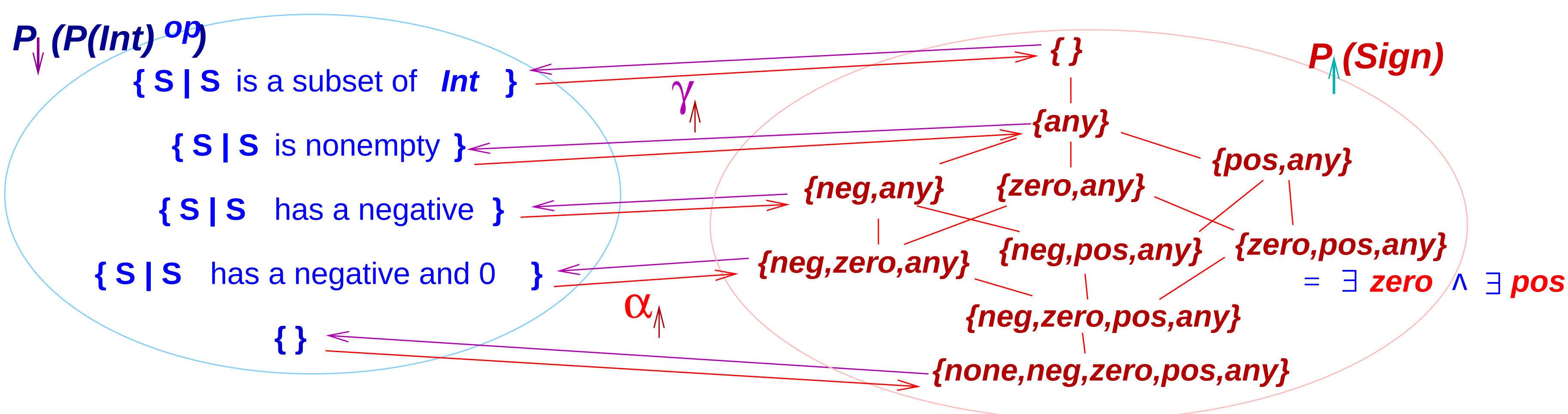
Theorem: For every $f^\sharp : A \to \mathcal{P}_\downarrow(A)$ and $T \in \mathcal{P}_\downarrow(A)$, $\mathrm{pre}_{f^\sharp}(T) \in \mathcal{P}_\uparrow(A)$!

Under*approximate* $f : C \to \mathcal{P}(C)$ ***by*** $f^{\flat} : A \to \mathcal{P}_{\uparrow}(A)$

Down-closed-set interpretation: $\downarrow\{zero, pos\}$ asserts $\forall\{zero, pos\} \equiv \forall(zero \vee pos)$ — all outputs are zero or positive :



Up-closed-set interpretation: $\uparrow\{zero, pos\}$ asserts $\exists\{zero, pos\}$ $\equiv \exists zero \wedge \exists pos$ — there exist 0 and a positive in the output:



Underapproximating $\mathrm{pre}_f(S) = \{c \mid f(c) \cap S \neq \emptyset\}$

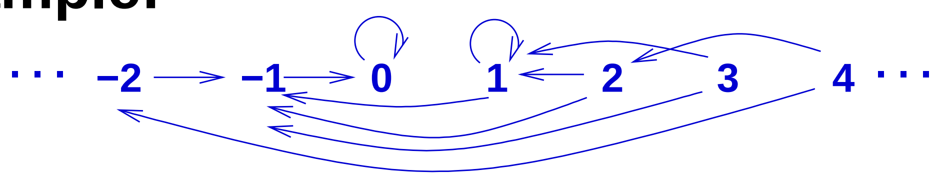
Use $\mathcal{P}_{\uparrow}(A)$ to define $f^{\flat}_{\mathrm{best}} : A \to \mathcal{P}_{\uparrow}(A)$ as

$$\begin{aligned} f^{\flat}_{\mathrm{best}}(a) &= (\alpha_{\uparrow} \circ (\{\!|\cdot|\!\} \circ f^{*}) \circ \gamma)(a) \\ &= \{a' \mid \forall c \in \gamma(a), f(c) \cap \gamma(a') \neq \emptyset\} \end{aligned}$$

and define $[\![\langle f \rangle \phi]\!]^{A} = \mathrm{pre}_{f^{\flat}_{\mathrm{best}}} [\![\phi]\!]^{A}$

Proposition: *(soundness)* $\mathrm{pre}_{f^{\flat}_{\mathrm{best}}}(T) \subseteq (\overline{\alpha_u} \circ \mathrm{pre}_f \circ \overline{\gamma})(T)$.

Example:

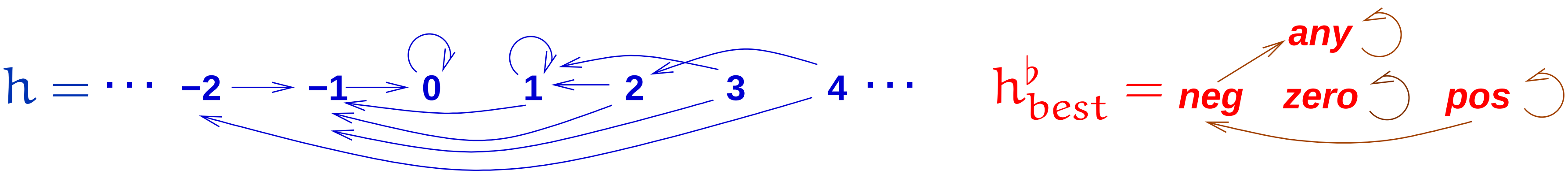
$h =$ 

$h^{\flat}_{\mathrm{best}} =$ 

We have $[\![\langle h \rangle(\mathit{neg} \vee \mathit{zero} \vee \mathit{pos})]\!] = \mathrm{pre}_h(\mathrm{Int}) = \mathrm{Int}$ and

$[\![\langle h \rangle(\mathit{neg} \vee \mathit{zero} \vee \mathit{pos})]\!]^{A} = \mathrm{pre}_{h^{\flat}_{\mathrm{best}}} \downarrow\{\mathit{neg}, \mathit{zero}, \mathit{pos}\} = \downarrow\{\mathit{zero}, \mathit{pos}\}$.

Improving precision with **focus**



For $\mathrm{pre}_h[\![\mathrm{neg} \vee zero \vee \mathrm{pos}]\!] = \mathrm{Int}$,

we lose precision: $\mathrm{pre}_{f^\flat_{best}}[\![\mathrm{neg} \vee zero \vee \mathrm{pos}]\!]^A = \downarrow\{zero, \mathrm{pos}\}$.

But $(\overline{\alpha_u} \circ \mathrm{pre}_f \circ \overline{\gamma})[\![\mathrm{neg} \vee zero \vee \mathrm{pos}]\!]^A_{ind} = \downarrow \mathrm{any} = Sign$!

Many analysis tools (e.g., TVLA [SagivRepsWilhelm02]) use a cases analysis, called focus*, to recover lost precision:*

$f^\flat_{best}(\mathrm{neg}) = \{\mathrm{any}\}$

$f^\flat_{best}(\mathrm{any}) = \{\mathrm{any}\}$

But any decomposes to the cases, neg, zero, pos. For each case, p, $p \in [\![\mathrm{neg} \vee zero \vee \mathrm{pos}]\!]^A$.

Theorem: When $\gamma : A \rightarrow \mathcal{P}(A)$ preserves joins, then $pre^{focus}_{f^\flat_{best}} = (\overline{\alpha_u} \circ \mathrm{pre}_f \circ \overline{\gamma})$.

Underapproximating post and $\widetilde{\text{post}}$

$$\text{post}_f(S) = f^*(S)$$
$$\widetilde{\text{post}}_f(S) = \{d \mid \forall c \in C, d \in f(c) \Rightarrow c \in S\}$$

Proposition: Let $f : D \rightarrow \mathcal{P}_\delta(D)$, where $\delta \in \{\downarrow, \uparrow\}$. Let $\tilde{\downarrow} = \uparrow$ and $\tilde{\uparrow} = \downarrow$. Then, for all $S \in \mathcal{P}(D)$,

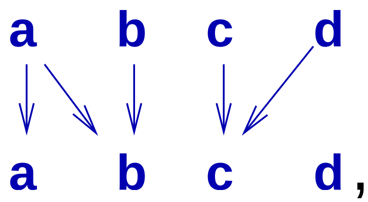
- $\widetilde{\text{pre}}_f(S) \in \mathcal{P}_\delta(D)$
- $\text{post}_f(S) \in \mathcal{P}_\delta(D)$
- $\text{pre}_f(S) \in \mathcal{P}_{\tilde{\delta}}(D)$
- $\widetilde{\text{post}}_f(S) \in \mathcal{P}_{\tilde{\delta}}(D)$.

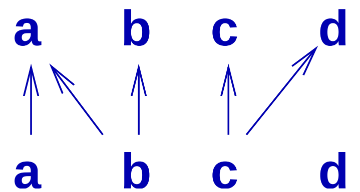
So, $\text{post}_{f^\flat} : A \rightarrow \mathcal{P}_\uparrow(A)$ *and* $\widetilde{\text{post}}_{f^\sharp} : A \rightarrow \mathcal{P}_\uparrow(A)$ *are* unsound.

Even worse, *there is no nontrivial overapproximating* $f^\sharp : A \rightarrow \mathcal{P}_\uparrow(A)$ to use with $\widetilde{\text{post}}$ because, for all $f^\sharp(a) \neq \emptyset$, upclosure implies that $\top_A \in f^\sharp(a)$, implying that $\overline{\gamma}(f^\sharp(a)) = C$. A similar problem arises for a nontrivial underapproximating $f^\flat : A \rightarrow \mathcal{P}_\downarrow(A)$.

What can we do ?

Solution: ***Invert*** $f : C \to \mathcal{P}(C)$ ***to*** $f^{-1} : C \to \mathcal{P}(C)$

If $f : C \to \mathcal{P}(C)$ is 

then $f^{-1} : C \to \mathcal{P}(C)$ is 

That is, $f^{-1}(c) = \{d \mid c \in f(d)\}$.

Proposition: [Loiseaux95] : $(f^{-1})^{-1} = f$, $\quad \mathrm{post}_f = \mathrm{pre}_{f^{-1}}$, and $\widetilde{\mathrm{post}}_f = \widetilde{\mathrm{pre}}_{f^{-1}}$.

Proposition: For $f : A \to \mathcal{P}_\delta(A)$, $\delta \in \{\downarrow, \uparrow\}$, $f^{-1} : A \to \mathcal{P}_{\tilde{\delta}}(A)$ is well defined and monotonic.

Underapproximating post_f *and* $\widetilde{\mathrm{post}}_f$

$$[\![\phi\overline{\langle f\rangle}]\!] \;=\; \mathrm{post}_f[\![\phi]\!] \;=\; \mathrm{pre}_{f^{-1}}[\![\phi]\!],$$

where $f : C \to \mathcal{P}(C)$

The inductively defined underapproximation is

$$[\![\phi\overline{\langle f\rangle}]\!]^A_{ind} = (\overline{\alpha_u} \circ \mathrm{pre}_{f^{-1}} \circ \overline{\gamma})[\![\phi]\!]^A.$$

This is soundly underapproximated by

$$[\![\phi\overline{\langle f\rangle}]\!]^A = \mathrm{pre}_{(f^{-1})^\flat_{best}}[\![\phi]\!]^A,$$

where $(f^{-1})^\flat_{best} : A \to \mathcal{P}_\uparrow(A)$

is $(f^{-1})^\flat_{best} = \overline{\alpha_\uparrow} \circ (\{\!|\cdot|\!\} \circ f^{-1})^* \circ \gamma.$

The same development applied to $\widetilde{\mathrm{post}}_f$ yields

$$[\![\phi\overline{[f]}]\!] = \widetilde{\mathrm{post}}_f[\![\phi]\!] = \widetilde{\mathrm{pre}}_{f^{-1}}[\![\phi]\!].$$

The most precise underapproximation is

$$[\![\phi\overline{\langle f\rangle}]\!]^{A}_{ind} = (\overline{\alpha_u} \circ \widetilde{\mathrm{pre}}_{f^{-1}} \circ \overline{\gamma})[\![\phi]\!]^{A}_{ind} = \widetilde{\mathrm{pre}}_{(f^{-1})^{\sharp}_{best}}[\![\phi]\!]^{A}_{ind},$$

where $(f^{-1})^{\sharp}_{best} : A \to \mathcal{P}_{\downarrow}(A)$

is $(f^{-1})^{\sharp}_{best} = \overline{\alpha_o} \circ (f^{-1})^{*} \circ \gamma.$

Computing abstract postconditions as preconditions of inverted state-transition relations is implemented in Steffen's fixpoint analysis machine [Steffen95] .

Summary

- We reviewed how to use *exact assertions* with an overapproximating Galois connection and how to apply *domain completions* to make assertions exact.
- When it is impractical to make assertions exact, we employed the *underapproximation Galois connection* on assertion sets.
- We proved that the forall-precondition transformer, $\widetilde{\mathrm{pre}}_f$, is best underapproximated by $\widetilde{\mathrm{pre}}_{f^\sharp_{best}}$.
- We used a *powerdomain of up-closed sets* to define $f^\flat_{best}$ and underapproximated pre_f by $\mathrm{pre}_{f^\flat_{best}}$.
- We formalized a *focussed* version of $\mathrm{pre}_{f^\flat_{best}}$ and proved it is the best approximation of pre_f when γ preserves joins.
- We inverted f to f^{-1} and applied the above machinery to underapproximate post_f and $\widetilde{\mathrm{post}}_f$.

References **This talk:** `www.cis.ksu.edu/~schmidt/papers`

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