Closed and logical relations for over- and under-approximation of powersets

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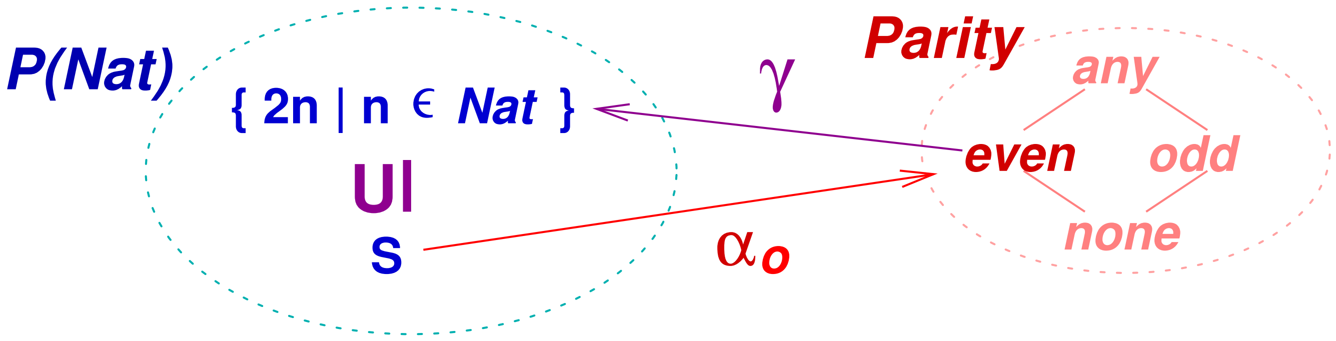
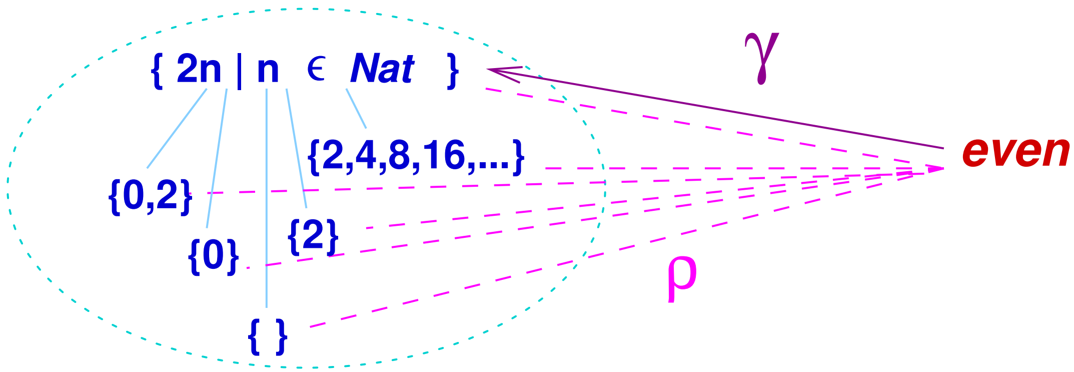
Background

Over-approximation *states a property of a program's outputs*



$even \in Parity$ asserts "$\forall even$" — all concrete outputs in set S are even-valued. (*We might write* $S\ \rho\ even$ *or* $S \models even$.)

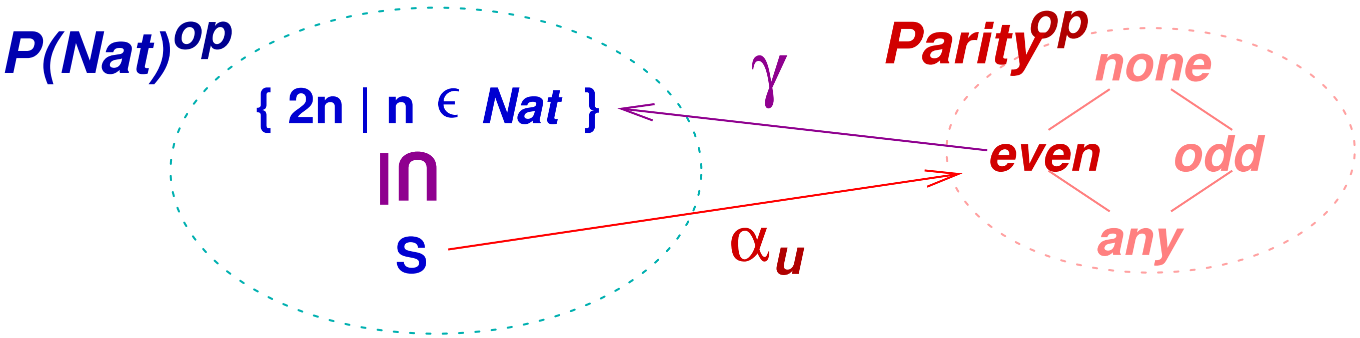
The upper adjoint, γ, selects the largest set approximated by $even$:



$$\gamma(a) = \cup\{S \mid S\ \rho\ a\}$$

Under-approximation *might be stated as the dual*

Here, $even$ assets that all evens are *included* in the concrete outputs:



This often abstracts constants to nothing, e.g., $[\![2]\!]^\flat_e = none$, where $\gamma(none) = \{\}$, because we require $\{2\} \supseteq \gamma(\alpha_u\{2\})$, forcing $\alpha_u\{2\} = none$.

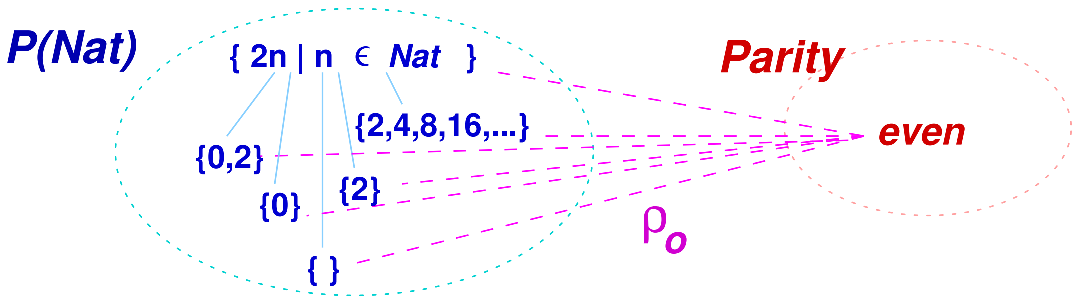
Thus, many program phrases are under-approximated to nothing:

$$[\![x+2]\!]^\flat_e = add^\flat([\![x]\!]^\flat_e, [\![2]\!]^\flat_e) = add^\flat(e(x), none) = none$$

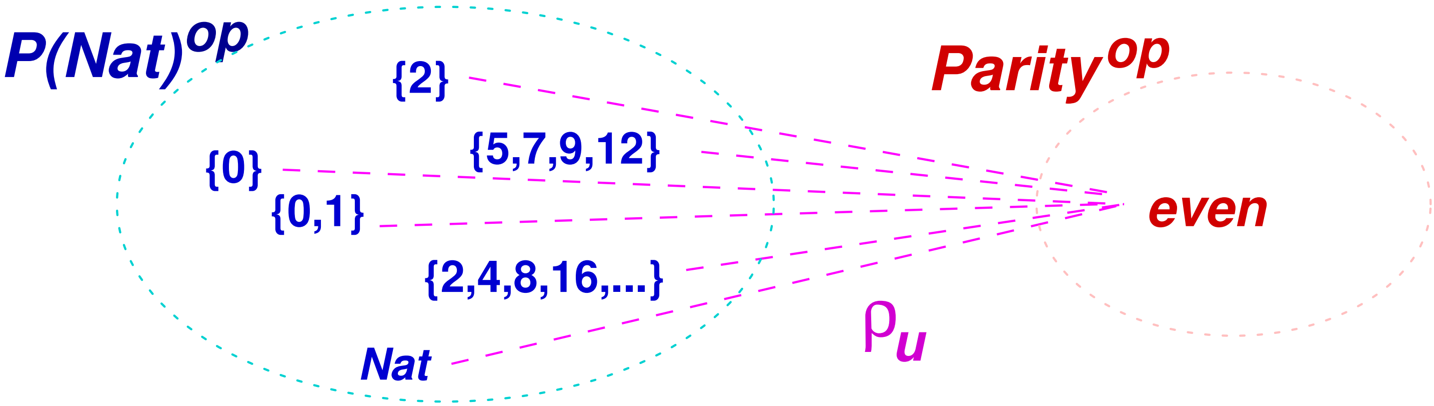
If we repair, say by including all constants, $n \in Nat$, in $Parity^{op}$, then to preserve $\gamma(\sqcap_{Parity^{op}} W) = \cup_{a \in W}\, \gamma(a)$, we must expand $Parity^{op}$ into $\mathcal{P}(Nat)^{op}$!

Under-approximation as existential quantification

If the over-approximating $\mathit{even} \in \mathrm{Parity}$ asserts "$\forall \mathit{even}$,"

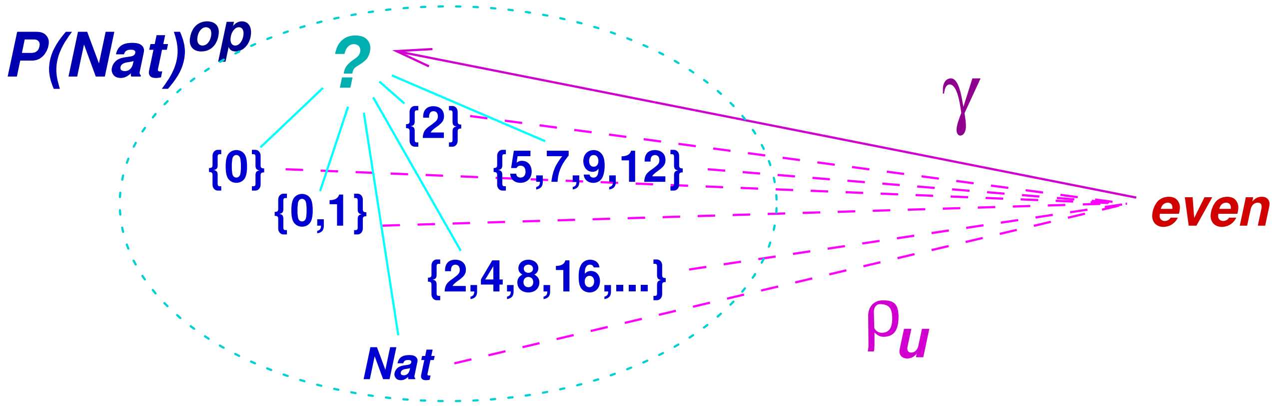


then the under-approximating $\mathit{even} \in \mathrm{Parity}^{op}$ *should assert* "$\exists \mathit{even}$"— there exists an even number in the program's outputs:



This provides a nontrivial under-approximation of constants, e.g., $[\![2]\!]^\flat_e = \mathit{even}$, and expressions: $[\![\mathtt{x} + 2]\!]^\flat_e = \mathrm{add}^\flat(e(\mathtt{x}), \mathit{even}) = e(\mathtt{x})$.

But we cannot define $\gamma : \mathrm{Parity}^{op} \rightarrow \mathcal{P}(\mathrm{Nat})^{op}$ in the usual way:



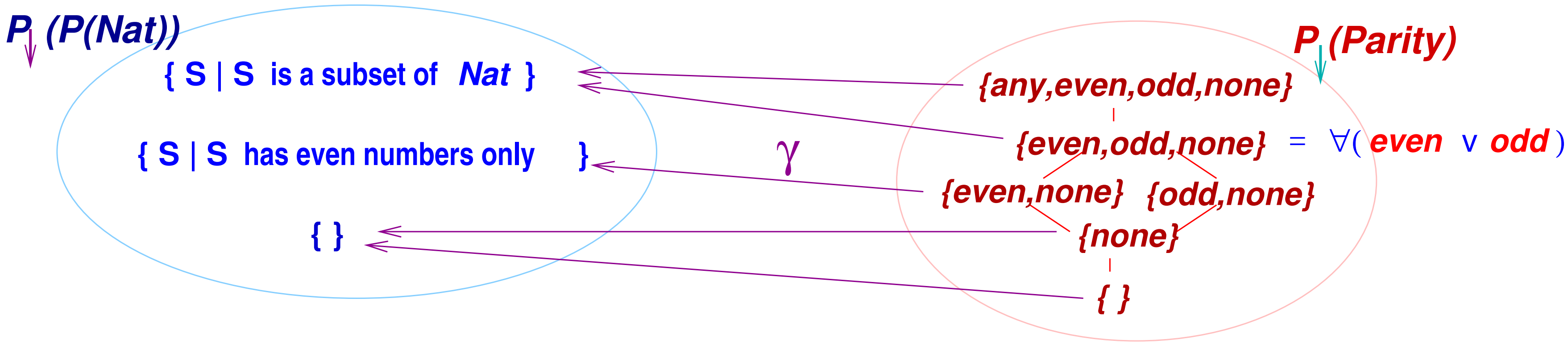
There is no best, minimal set that contains an even number.

Indeed, *even*'s concretization is not a single set — it is a *set of sets*:

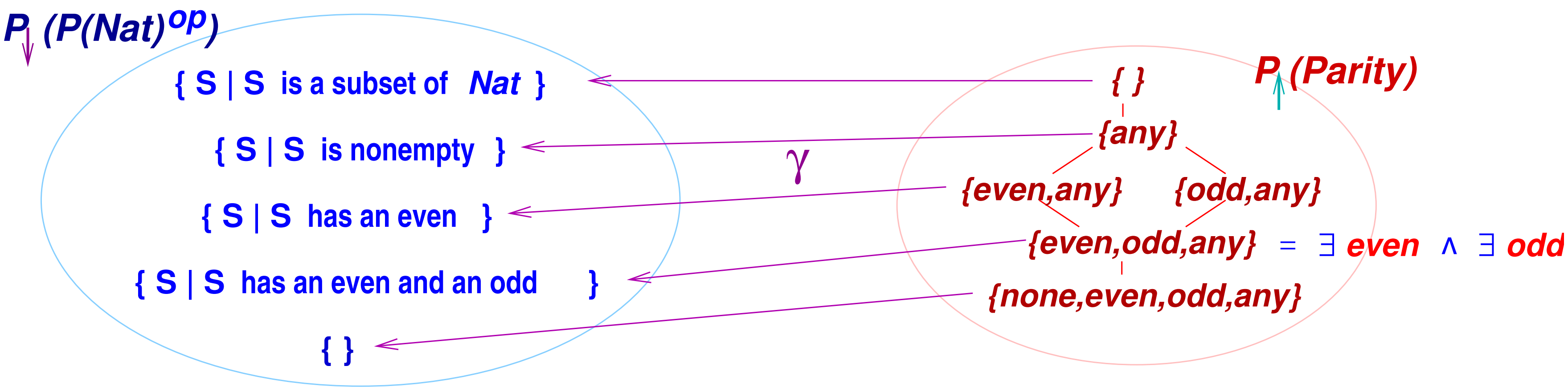
$$\gamma(\mathit{even}) = \{S \in \mathcal{P}(\mathrm{Nat})^{op} \mid S \; \rho_u \; \mathit{even}\}$$

This suggests that we work with *power-domains* in both the concrete and abstract domains.

Universal (over-approximating) interpretation: $\{\mathrm{even}, \mathrm{odd}\}$ asserts $\forall\{\mathrm{even}, \mathrm{odd}\} \equiv \forall(\mathrm{even} \vee \mathrm{odd})$ — all outputs are even- or odd-valued: Use a lower power-domain (lower sets) for the abstract domain.



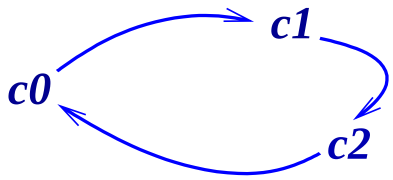
Existential (under-approximating) interpretation: $\{\mathrm{even}, \mathrm{odd}\}$ asserts $\exists\{\mathrm{even}, \mathrm{odd}\} \equiv \exists\mathrm{even} \wedge \exists\mathrm{odd}$ — there exists an even-valued and an odd-valued output: Use an upper power-domain (upper sets).



Dennis Dams's **mixed transition systems** *employ the universal and existential abstractions*

A transition system: $\Sigma = \{c_0, c_1, c_2\}$

$R = \{(c_0, c_1), (c_1, c_2), (c_2, c_0)\}$



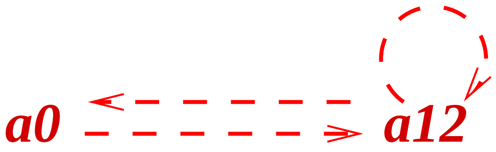
Approximating the states: Note: $\perp$ and $\top$ omitted for brevity.

$$\alpha\{c_0\} = a_0, \qquad \alpha\{c_1\} = a_{12} = \alpha\{c_2\} = \alpha\{c_1, c_2\}$$

That is, $c_0 \; \rho \; a_0$, $c_1 \; \rho \; a_{12}$, and $c_2 \; \rho \; a_{12}$.

Over-approximation transitions ("*may*"):

$$R^\sharp = \{(a_0, a_{12}), (a_{12}, a_{12}), (a_{12}, a_0)\}$$



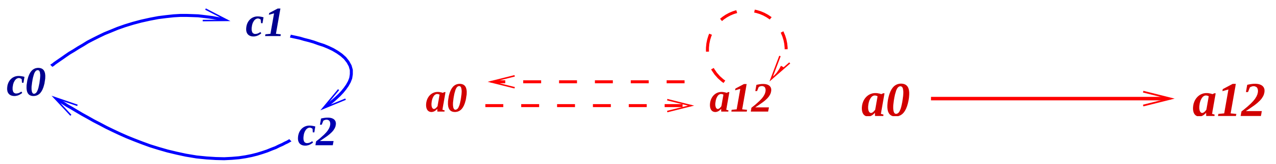
As a function, $R^\sharp(a_0) = \{a_{12}\} \equiv \forall a_{12}$ and $R^\sharp(a_{12}) = \{a_0, a_{12}\} \equiv \forall(a_0 \vee a_{12})$.

Under-approximation transitions ("*must*"):

$$R^\flat = \{(a_0, a_{12})\}$$

As a function, $R^\flat(a_0) = \{a_{12}\} \equiv \exists a_{12}$ and $R^\flat(a_{12}) = \{\} \equiv \exists() \equiv \perp$

The *mixed transition system* is $(\{a_0, a_{12}\}, R^\flat, R^\sharp)$.



From Galois connection, $\mathcal{P}(C)\langle\alpha,\gamma\rangle A$, Dams defines this *simulation relation*: $c\ \rho\ a$ *iff* $c \in \gamma(a)$. From $R \subseteq C \times C$, he defines

$$aR^{\sharp}a' \text{ iff } a' \in \{\alpha(Y) \mid Y \in \min\{S' \mid R^{\exists\exists}(\gamma(a), S')\}\}$$
$$aR^{\flat}a' \text{ iff } a' \in \{\alpha(Y) \mid Y \in \min\{S' \mid R^{\forall\exists}(\gamma(a), S')\}\}$$

and he proves $R \lhd_{\rho} R^{\sharp}$, that is, $R^{\sharp}$ ρ*-simulates* R
$R^{\flat} \lhd_{\rho^{-1}} R$, that is, $R^{\flat}$ *is* ρ*-simulated by* R

This gives him **soundness** for $\Box$ $(\forall R)$ and $\Diamond$ $(\exists R)$:

Define $a \models \Box\phi$ *iff for all* a', $aR^{\sharp}a'$ *implies* $a' \models \phi$
$a \models \Diamond\phi$ *iff there exists* a' *such that* $aR^{\flat}a'$ *and* $a' \models \phi$

Then, $a \models \phi$ **and** $c\ \rho\ a$ **imply** $c \models \phi$**.**

And with lots of hard work, he proves **best precision**: For all ρ-, ρ^{-1}-simulations of R, $R^{\sharp}$ and $R^{\flat}$ preserve the *most* $\Box\Diamond$-properties.

Can we derive $R^\sharp$ and $R^\flat$ and prove soundness and precision directly from Galois-connection theory?

Yes — we treat $R \subseteq C \times C$ as $R : C \to \mathcal{P}(C)$.
Then, we have $R^\sharp : A \to \mathcal{P}_L(A)$, where $\mathcal{P}_L(\cdot)$ is a *lower powerset* ($\subseteq$) constructor.

We "lift" the Galois connection, $\mathcal{P}(C)\langle\alpha_\tau, \gamma_\tau\rangle A$, on the states to a Galois connection on powersets, $F[\mathcal{P}(C)]\langle\alpha_{F[\tau]}, \gamma_{F[\tau]}\rangle\mathcal{P}_L(A)$, so that

1. *$R^\sharp$ ρ-simulates R iff* $\quad ext_{F[\tau]}(R) \circ \gamma_\tau \sqsubseteq_{A \to F[\mathcal{P}(C)]} \gamma_{F[\tau]} \circ R^\sharp$
2. *the soundness of $a \models \Box\phi$ follows from Item 1*
3. $R^\sharp{}_{best} = \alpha_{F[\tau]} \circ ext_{F[\tau]}(R) \circ \gamma_\tau$

We do similar work for $R^\flat{}_{best} : A \to \mathcal{P}_U(A)$ and $\Diamond\phi$, where $\mathcal{P}_U(\cdot)$ is an *upper ($\supseteq$-ordered) powerset* constructor.

For over-approximation, we can use $F = id$ or $F = \mathcal{P}_L(\cdot)$; for under-approximation, we must use $F = \mathcal{P}_L(\cdot^{op})$. (We will see why....)

Our results from reworking Dams's constructions

1. Starting from approximation relations, $\rho \subseteq C \times A$, we generate Galois connections from *U-GLB-L-LUB-closed* relations cf. [*Mycroft-Jones 86, Cousot-Cousot JLC 92*].

2. We define *lower and upper powerset constructions*, weaker forms of powerdomain but strong enough for abstraction studies. They are the *join completions* of [*Cousot-Cousot ICCL 94*].

3. We use powerset types in a family of *logical relations*, show how the family preserves the closure properties in 1., and prove that a simulation proof is an instance of proof via logical relations. We obtain Dams's most-precise simulation results "for free."

4. We extract *validation and refutation logics* from the logical relations, state their resemblance to Hennessy-Milner logic (and description logic), and obtain easy proofs of soundness.

Closed approximation relations

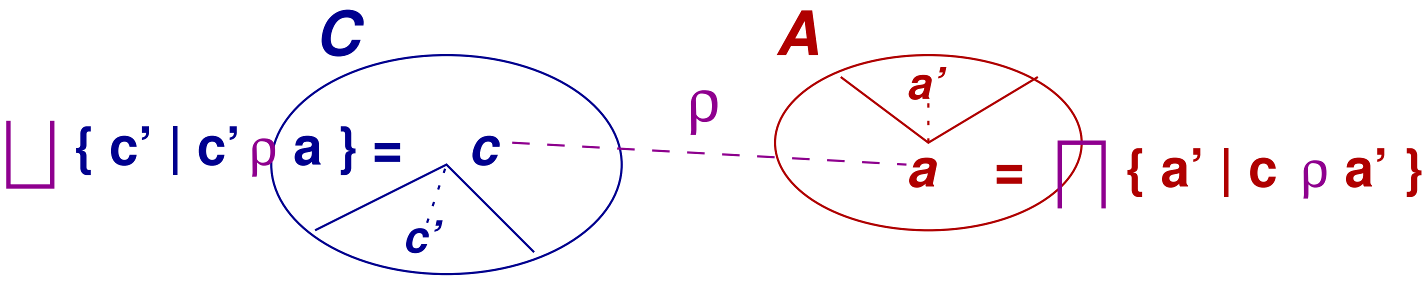
Closed relations and Galois connections

Let C and A be complete lattices, and let $\rho \subseteq C \times A$.

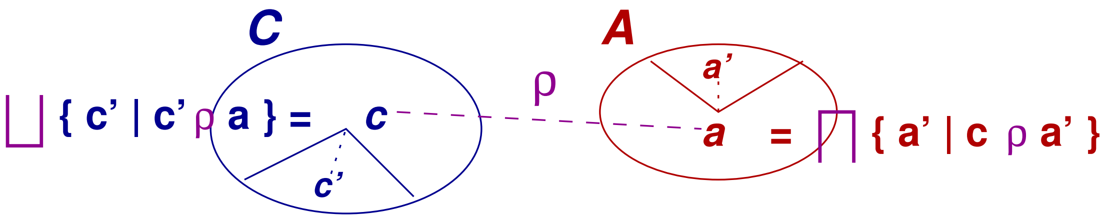
$c \; \rho \; a$ means that c is modelled/approximated by a

Definition: For all $c, c' \in C$, $a, a' \in A$, for $\rho \subseteq C \times A$, ρ is

1. *U-closed* iff $c \; \rho \; a$, $a \sqsubseteq a'$ imply $c \; \rho \; a'$
2. *GLB-closed* iff $c \; \rho \; \sqcap \{a \mid c \; \rho \; a\}$
3. *L-closed* iff $c \; \rho \; a$, $c' \sqsubseteq c$ imply $c' \; \rho \; a$
4. *LUB-closed* iff $\sqcup \{c \mid c \; \rho \; a\} \; \rho \; a$



Origins: Hartmanis and Stearns 1964 (pair algebras); Mycroft-Jones 1986 (LU-closure); Cousot-Cousot JLC 1992; Backhouse-Backhouse 1998



Proposition: For L-LUB-U-GLB-closed $\rho \subseteq C \times A$, $C\langle \alpha_\rho, \gamma_\rho \rangle A$ is a Galois connection, where

- $\alpha_\rho(c) = \sqcap\{a \mid c \;\rho\; a\}$
- $\gamma_\rho(a) = \sqcup\{c \mid c \;\rho\; a\}$

Intuition: U-closed makes γ_ρ mono; L-closed makes α_ρ mono; GLB-closed ensures α_ρ selects the most precise sound answer; LUB-closed ensures γ_ρ selects the most general sound answer.

Note that $c \;\rho\; a$ iff $c \sqsubseteq_C \gamma_\rho a$ iff $\alpha_\rho c \sqsubseteq_A a$.

Proposition: For Galois connection, $C\langle \alpha, \gamma \rangle A$, define $\rho_{\alpha\gamma} \subseteq C \times A$ as $\{(c, a) \mid \alpha c \sqsubseteq a\}$. Then,

$$\rho_{\alpha\gamma} \text{ is L-LUB-U-GLB-closed and } \langle \alpha_{\rho_{\alpha\gamma}}, \gamma_{\rho_{\alpha\gamma}} \rangle = \langle \alpha, \gamma \rangle.$$

"Completing" U-GLB-closed $\rho \subseteq C \times A$ into a Galois connection between $\mathcal{P}(C)$ and A

Here is a standard technique: Let C be a (discretely ordered) set and let A be a complete lattice.

For $\rho \subseteq C \times A$, define $\bar{\rho} \subseteq \mathcal{P}(C) \times A$ as $S\ \bar{\rho}\ a$ *iff for all* $c \in S$, $c\ \rho\ a$.

Theorem: If ρ is U-GLB-closed, then $\bar{\rho}$ is L-LUB-U-GLB-closed, and $\mathcal{P}(C)\langle \alpha_{\bar{\rho}}, \gamma_{\bar{\rho}} \rangle A$ is a Galois connection, where $\gamma_{\bar{\rho}}\, a = \sqcup\{S \mid S\ \bar{\rho}\ a\} = \{c \mid c\ \rho\ a\}$.

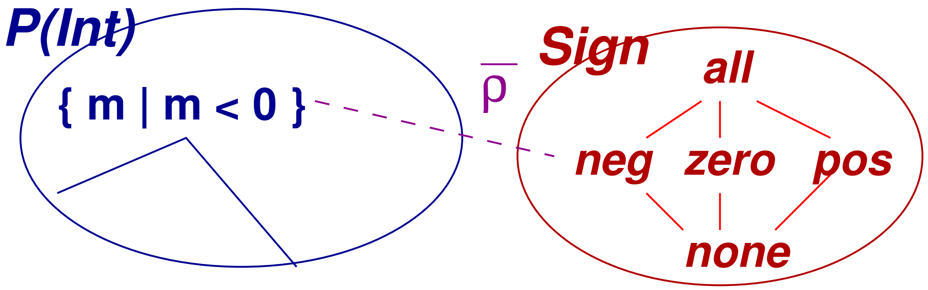
Example: Let Int be the discretely ordered set of integers: $\rho \subseteq \mathrm{Int} \times \mathrm{Sign}$

$-n\ \rho\ \mathrm{neg}$

$0\ \rho\ \mathrm{zero}$

$+n\ \rho\ \mathrm{pos}$

$n\ \rho\ \mathrm{all}$



ρ is L-U-GLB-closed but not LUB-closed. It is completed to $\bar{\rho} \subseteq \mathcal{P}(\mathrm{Int}) \times \mathrm{Sign}$.

Powersets

Lower powersets

D's "powerset" is a complete lattice, E, with monotone *singleton* and *union* operations: $PD = (E, \sqsubseteq_E, \{\!|\cdot|\!\} : D \to E, \uplus : E \times E \to E)$.
Define *membership* as $c \tilde{\in} S$ iff $\{\!|c|\!\} \uplus S = S$.

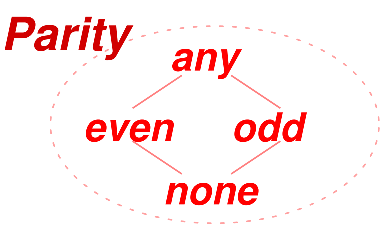
A *lower powerset*, $\mathcal{P}_L(D)$, treats $\sqsubseteq_E$ as $\subseteq$: For all $S_1, S_2 \in E$,

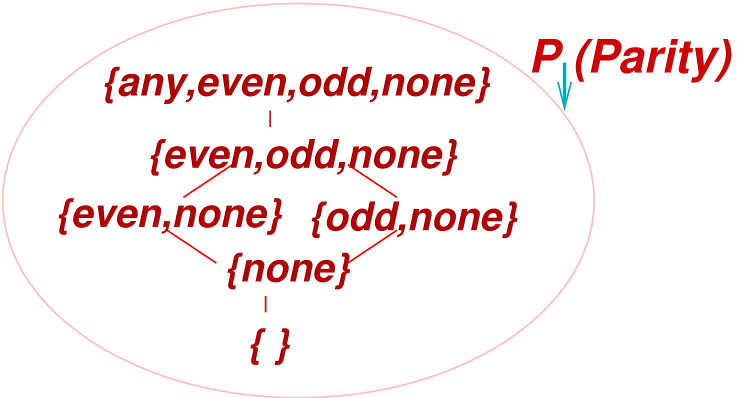
$S_1 \sqsubseteq_E S_2$ iff (for all $x \tilde{\in} S_1$, there exists $y \tilde{\in} S_2$ such that $x \sqsubseteq_D y$)

Down-set (order-ideal) completion: For $d \in D$, $S \subseteq D$, define
$\downarrow d = \{e \in D \mid e \sqsubseteq_D d\}$ and $\downarrow S = \cup\{\downarrow d \mid d \in S\}$.

Define $\mathcal{P}_{\downarrow}(D) = (\{\downarrow S \mid S \subseteq D\}, \subseteq, \downarrow, \cup)$ — all down-closed subsets of D

Example:





Upper powersets

There is a dual construction:

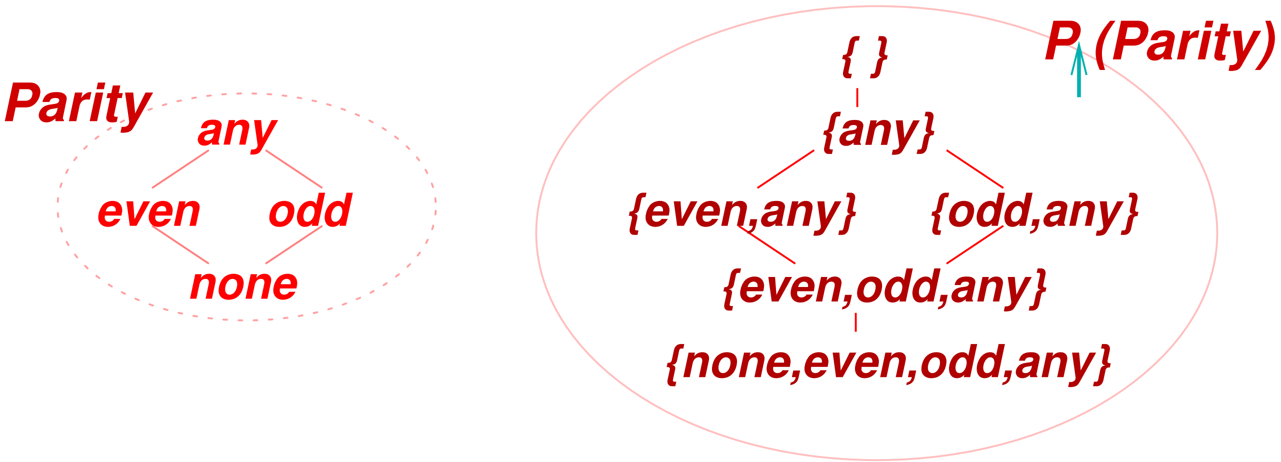
An *upper powerset*, $\mathcal{P}_U(D)$, treats $\sqsubseteq_E$ as $\supseteq$: For all $S_1, S_2 \in E$,

$S_1 \sqsubseteq_E S_2$ iff (for all $y \tilde{\in} S_2$, there exists $x \tilde{\in} S_1$ such that $x \sqsubseteq_D y$)

Up-set (filter) completion: For $d \in D$ and $S \subseteq D$, define
$\uparrow d = \{e \in D \mid d \sqsubseteq_D e\}$ and $\uparrow S = \cup\{\uparrow d \mid d \in S\}$.

Define $\mathcal{P}_{\uparrow}(D) = (\{\uparrow S \mid S \subseteq D\}, \supseteq, \uparrow, \cup)$ — all up-closed subsets of D

Example:



Logical relations

Logical relations

We now attach typings to the relations. Here are the types:

$$\tau \ ::= \ b \mid \tau_1 \rightarrow \tau_2 \mid \mathcal{P}_L(\tau) \mid \mathcal{P}_U(\tau) \mid \bar{\tau}$$

$\bar{\tau}$ is a special case of $\mathcal{P}_L(\tau)$ and names the completion of U-GLB-closed $\rho \subseteq C \times A$ to $\bar{\rho} \subseteq \mathcal{P}(C) \times A$, seen earlier

Let C_τ and A_τ be p.o.-sets of the appropriate form (e.g., $A_{\tau_1 \rightarrow \tau_2}$ is a lattice of monotone functions, $A_{\mathcal{P}_U(\tau)}$ is an upper powerset, etc.)

We define this family of logical relations, $\rho_\tau \subseteq C_\tau \times A_\tau$*:*

ρ_b is given

$f \ \rho_{\tau_1 \rightarrow \tau_2} \ f^\sharp$ iff for all $c \in C_{\tau_1}, a \in A_{\tau_1}, c \ \rho_{\tau_1} \ a$ implies $f(c) \ \rho_{\tau_2} \ f^\sharp(a)$

$S \ \rho_{\mathcal{P}_L(\tau)} \ T$ iff for all $c \tilde{\in} S,$ there exists $a \tilde{\in} T$ such that $c \ \rho_\tau \ a$

$S \ \rho_{\mathcal{P}_U(\tau)} \ T$ iff for all $a \tilde{\in} T,$ there exists $c \tilde{\in} S$ such that $c \ \rho_\tau \ a$

$S \ \rho_{\bar{\tau}} \ a$ iff for all $c \tilde{\in} S, c \ \rho_\tau \ a$. *That is,* $S \ \rho_{\bar{\tau}} \ a$ iff $S \ \rho_{\mathcal{P}_L(\tau)} \ \{\!|a|\!\}$

Simulation relations are logical relations

Binary relations are the key component in simulation proofs:

For $\rho \subseteq C \times A$, transition relations, $R \subseteq C \times C$, $R^\sharp \subseteq A \times A$,

Definition: $R^\sharp$ *simulates* R, written $R \lhd_\rho R^\sharp$, iff for all $c, c' \in C$ and $a \in A$,

$c \; \rho \; a$ and $c \; R \; c'$ imply there exists $a' \in A$ s.t. $a \; R^\sharp \; a'$ and $c' \; \rho \; a'$.

Say that we represent R and $R^\sharp$ as multi-functions, $R : C \rightarrow \mathcal{P}_L(C)$ and $R^\sharp : A \rightarrow \mathcal{P}_L(A)$:

Theorem:

1. $R \lhd_{\rho_b} R^\sharp$ iff $R \; \rho_{b \rightarrow \mathcal{P}_L(b)} \; R^\sharp$
2. $R^\flat \lhd_{\rho_b^{-1}} R$ iff $R \; \rho_{b \rightarrow \mathcal{P}_U(b)} \; R^\flat$

Closure properties of logical relations

$f\ \rho_{\tau_1 \to \tau_2}\ f^\sharp$ iff for all $c \in C_{\tau_1}$, $a \in A_{\tau_1}$, $c\ \rho_{\tau_1}\ a$ implies $f(c)\ \rho_{\tau_2}\ f^\sharp(a)$

$S\ \rho_{\mathcal{P}_L(\tau)}\ T$ iff for all $c \tilde{\in} S$, there exists $a \tilde{\in} T$ such that $c\ \rho_\tau\ a$

$S\ \rho_{\mathcal{P}_U(\tau)}\ T$ iff for all $a \tilde{\in} T$, there exists $c \tilde{\in} S$ such that $c\ \rho_\tau\ a$

$S\ \rho_{\bar{\tau}}\ a$ iff for all $c \tilde{\in} S$, $c\ \rho_\tau\ a$

Theorem: For $\rho_\tau \subseteq C_\tau \times A_\tau$ and for $F[\tau] \in \{\tau' \to \tau,\ \mathcal{P}_L(\tau),\ \mathcal{P}_U(\tau),\ \bar{\tau}\}$,

If ρ_τ is L-closed, then so is $\rho_{F[\tau]}$.

If ρ_τ is U-closed, then so is $\rho_{F[\tau]}$.

If ρ_τ is U-GLB-closed, then so are $\rho_{\tau' \to \tau}$, $\rho_{\bar{\tau}}$, and $\rho_{\mathcal{P}_L(\tau)}$.

If ρ_τ is L-LUB-closed, then so are $\rho_{\tau' \to \tau}$ and $\rho_{\mathcal{P}_U(\tau)}$.

Proposition: $\rho_{\bar{\tau}}$ and $\rho_{\mathcal{P}_L(\tau)}$ are always L-closed, and $\rho_{\mathcal{P}_U(\tau)}$ is always U-closed.

Alas, LUB-closure is not guaranteed for $\mathcal{P}_{\mathrm{L}}(\tau)$ and neither is GLB-closure for $\mathcal{P}_{\mathrm{U}}(\tau)$.

But there are some sufficient conditions upon the choice of lower- and upper-powerset that ensure these closures.

Here are two simple but useful examples:

Proposition: $\rho_{\mathcal{P}_{\mathrm{L}}(\tau)} \subseteq \mathcal{P}_{\downarrow}(C_\tau) \times \mathcal{P}_{\mathrm{L}}(A_\tau)$ is always LUB-closed.

Proposition: $\rho_{\mathcal{P}_{\mathrm{U}}(\tau)} \subseteq \mathcal{P}_{\mathrm{U}}(C_\tau) \times \mathcal{P}_{\uparrow}(A_\tau)$ is aways GLB-closed.

Dams's results

Synthesizing a most-precise simulation

Dams proved, for $\mathcal{P}(C)\langle\alpha,\gamma\rangle A$ and transition relation $R \subseteq C \times C$, that the most-precise, sound abstract relation $R_0 \subseteq A \times A$ is

$$R_0(a, a') \text{ iff } a' \in \{\alpha(Y) \mid Y \in \min\{S' \mid R^{\exists\exists}(\gamma(a), S')\}\}$$

Reformatted as a function, this reads

$$R_0(a) = \{\alpha(s') \mid \exists s \in \gamma(a), s' \in R(s)\}$$

We can derive Dams's result: Given U-GLB-closed $\rho_b \subseteq C \times A$ and transition function $R : C \to \mathcal{P}(C)$, we derive $R_0 : A \to \mathcal{P}_\downarrow(A)$:

1. *We use the closure properties to generate L-LUB-U-GLB-closed relations,* $\rho_{\bar{b}} \subseteq \mathcal{P}(C) \times A$ *and* $\rho_{\mathcal{P}_L(b)} \subseteq \mathcal{P}(C) \times \mathcal{P}_\downarrow(A)$*.*
2. *We synthesize* $R^\sharp{}_{best} : A \to \mathcal{P}_\downarrow(A)$ *in the expected way:*

$$R^\sharp{}_{best} = \alpha_{\rho_{\mathcal{P}_L(b)}} \circ ext(R) \circ \gamma_{\rho_{\bar{b}}} = R_0$$

Synthesizing a most-precise **dual** *simulation*

Dams proved, for $\mathcal{P}(C)\langle\alpha,\gamma\rangle A$ and $R \subseteq C \times C$, that the best underapproximating relation $R_1 \subseteq A \times A$ is

$$R_1(a,a') \text{ iff } a' \in \{\alpha(Y) \mid Y \in \min\{S' \mid R^{\forall\exists}(\gamma(a),S')\}\}$$

Reformatted as a function, this reads

$$R_1(a) = \{\alpha(Y) \mid Y \in \min\{S' \mid \text{for all } s \in \gamma(a), R(s) \cap S' \neq \{\}\}\}$$

We must work a bit harder, but we can derive the same result:

Given $R : C \to \mathcal{P}(C)$ and U-GLB-closed $\rho_b \subseteq C \times A$, we derive $R_1 : A \to \mathcal{P}_{\uparrow}(A)$....

We derive $R_1 : A \rightarrow \mathcal{P}_{\uparrow}(A)$**:**

1. *We generate L-LUB-U-GLB-closed* $\rho_{\bar{b}} \subseteq \mathcal{P}(C) \times A$
2. *We generate* $\rho_{\bar{\mathcal{P}}_{U}(b)} \subseteq \mathcal{P}_{\downarrow}(\mathcal{P}(C)^{op}) \times \mathcal{P}_{\uparrow}(A)$ *in stages:*
   (a) *begin with U-GLB-closed* $\rho_b \subseteq C \times A$ (because C is discretely ordered, ρ_b is L-closed also);
   (b) *lift to sets of answers*: lift the relation to L-U-GLB-closed $\rho_{\mathcal{P}_U(\tau)} \subseteq \mathcal{P}(C)^{op} \times \mathcal{P}_{\uparrow}(A)$;
   (c) *introduce LUB-closure (giving a Galois connection):* complete the relation to $\rho_{\bar{\mathcal{P}}_U(\tau)} \subseteq \mathcal{P}_{\downarrow}(\mathcal{P}(C)^{op}) \times \mathcal{P}_{\uparrow}(A)$.
3. *We synthesize* $R^{\flat}{}_{best} : A \rightarrow \mathcal{P}_{\uparrow}(A)$:

$$R^{\flat}{}_{best} = \alpha_{\rho_{\bar{\mathcal{P}}_U(b)}} \circ ext(\{\!\!\{ \cdot \}\!\!\} \circ R) \circ \gamma_{\rho_{\bar{b}}} = R_1$$

where $ext(\{\!\!\{ \cdot \}\!\!\} \circ R) : \mathcal{P}(C)^{op} \rightarrow \mathcal{P}_{\downarrow}(\mathcal{P}(C)^{op})$ maps a set of concrete arguments to the set of R-successor sets of the arguments.

As seen in the talk's introductory example, the relation in (b) lacks LUB-closure.

Validation and refutation logics

A logic generated from the logical relations

We define this language of assertions,

$$\phi ::= p_b \mid f.\phi \mid \forall\phi \mid \exists\phi$$

and this semantics of typed judgements for both concrete domains, C_τ, and abstract domains, A_τ:

$d \models_b p_b$ is given, for $d \in D_b$

$d \models_{\tau_1 \to \tau_2} f.\phi$ if $f(d) \models_{\tau_2} \phi$, for $d \in D_{\tau_1}, f \in D_{\tau_1 \to \tau_2}$

$S \models_{\mathcal{P}_L(\tau)} \forall\phi$ if for all $d \tilde{\in} S, d \models_\tau \phi$, for $S \in D_{\mathcal{P}_L(\tau)}$

$S \models_{\mathcal{P}_U(\tau)} \exists\phi$ if there exists $d \tilde{\in} S$ such that $d \models_\tau \phi$, for $S \in D_{\mathcal{P}_U(\tau)}$

The judgement form for $\bar{\tau}$ is a special case of $\mathcal{P}_L(\tau)$'s:

$$S \models_{\bar{\tau}} \phi \text{ if } c \models_\tau \phi, \text{ for all } c \in S, S \in \mathcal{P}_L(C_\tau)$$
$$a \models_{\bar{\tau}} \phi \text{ if } a \models_\tau \phi, \text{ for } a \in A_\tau$$

Some "syntactic sugar":

$$d \models \forall R\phi \text{ (that is, } d \models \Box\phi) \quad \text{abbreviates} \quad d \models_{\tau_1 \to \mathcal{P}_L(\tau_2)} R.\forall\phi$$

$$d \models \exists R\phi \ (d \models \Diamond\phi) \quad \text{abbreviates} \quad d \models_{\tau_1 \to \mathcal{P}_U(\tau_2)} R.\exists\phi$$

This reveals that the logic extracted from the logical relations is a variant of *Hennessy-Milner logic* or *description logic* or *branching-time temporal logic*.

$\tau ::= b \mid \tau_1 \to \tau_2 \mid \mathcal{P}_L(\tau) \mid \mathcal{P}_U(\tau) \mid \bar{\tau}$

Assume, for all function symbols, f, typed $\tau_1 \to \tau_2$, there are interpretations $f : C_{\tau_1} \to C_{\tau_2}$, and $f^\sharp : A_{\tau_1} \to A_{\tau_2}$, such that $f \; \rho_{\tau_1 \to \tau_2} \; f^\sharp$. Also, we formalize when judgements $a \models_\tau \phi$ are *well formed*.

Definition: $\models_\tau \phi$ *is* ρ_τ*-sound* iff for all $c \in C_{\tau_1}$, $a \in A_{\tau_2}$,

$a \models_\tau \phi$ is well formed, holds true, and $c \; \rho_\tau \; a$ imply $c \models_\tau \phi$.

Assume that all $\models_b p$ are ρ_b-sound.

Theorem: For all types, τ, we have that $\models_\tau \phi$ are ρ_τ-sound.

We can add the logical connectives,

$$d \models_\tau \phi_1 \wedge \phi_2 \text{ if } d \models_\tau \phi_1 \text{ and } d \models_\tau \phi_2$$

$$d \models_\tau \phi_1 \vee \phi_2 \text{ if } d \models_\tau \phi_1 \text{ or } d \models_\tau \phi_2$$

and prove these ρ_τ-sound as well.

Validating $\neg\phi$ *requires a* refutation logic

Define $c \models_\tau \neg\phi$ *iff* $c \not\models_\tau \phi$.

We have a logic that validates ϕ for $c \in C$ by validating it for $a \in A$, so we might have also a logic that *refutes* properties similarly:

Read $a \models_\tau^{\neg pos} \phi$ as "it is not possible that any value modelled by a has property ϕ."

$a \models_b^{\neg pos} p$ is given, for $a \in A_b$

$a \models_{\tau_1 \to \tau_2}^{\neg pos} f.\phi$ if $f(a) \models_{\tau_2}^{\neg pos} \phi$, for $a \in A_{\tau_1}$, $f \in A_{\tau_1 \to \tau_2}$

$T \models_{\mathcal{P}_U(\tau)}^{\neg pos} \forall\phi$ if exists $a \tilde{\in} T$, $a \models_\tau^{\neg pos} \phi$, for $T \in A_{\mathcal{P}_U(\tau)}$

$T \models_{\mathcal{P}_L(\tau)}^{\neg pos} \exists\phi$ if for all $a \tilde{\in} T$, $a \models_\tau^{\neg pos} \phi$, for $T \in A_{\mathcal{P}_L(\tau)}$

$a \models_{\overline{\tau}}^{\neg pos} \phi$ if $a \models_\tau^{\neg pos} \phi$, for $a \in A_\tau$

Definition: $\models_\tau^{\neg pos} \phi$ is ρ_τ-*sound* iff for all $c \in C_{\tau_1}$, $a \in A_{\tau_2}$, $a \models_\tau^{\neg pos} \phi$ is well formed, holds, and $c \; \rho_\tau \; a$ imply $c \not\models_\tau \phi$.

Theorem: All $\models_\tau^{\neg pos} \phi$ are ρ_τ-sound.

The case for $\models_{\bar{\tau}}^{\neg pos}\ \phi$ shows significant loss of precision: $a \models_{\bar{\tau}}^{\neg pos}\ \phi$ and $S\ \rho_{\bar{\tau}}\ a$ imply *for all* $c \in S$, that $c \models_{\tau}^{\neg pos}\ \phi$, whereas we need only show that *there exists* some $c \in S$, such that $c{\models^{\neg pos}}_{\tau}\phi$.

Corollary: $a \models_{\tau} \neg\phi$ *if* $a \models_{\tau}^{\neg pos}\ \phi$ is sound for ρ_{τ}.

$a \models_{\tau}^{\neg pos} \neg\phi$ *if* $a \models_{\tau} \phi$ is sound for ρ_{τ}.

(i) In the refutation logic, $\models_{\tau}^{\neg pos}\ \phi$, the roles of $\mathcal{P}_{\mathrm{L}}(\tau)$ and $\mathcal{P}_{\mathrm{U}}(\tau)$ are exchanged. This, as well as the need to validate a mix of $\forall$ and $\exists$, means we must employ $\mathrm{R}^{\sharp}$ and $\mathrm{R}^{\flat}$ to validate/refute assertions — this is the idea behind mixed/modal transition systems.

(ii) The Sagiv-Reps-Wilhelm TVLA system simultaneously calculates validation and refutation logics.

(iii) We might approximate every concrete set by a *pair* of lower and upper approximations: $\rho_{\mathrm{P}\tau} \subseteq \mathrm{PC} \times (\mathcal{P}_{\mathrm{L}}(A) \times \mathcal{P}_{\mathrm{U}}(A))$. This motivates sandwich- and mixed-powerdomains for over-under-approximation of sets [Huth-Jagadeesan-Schmidt].

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