Comparing completeness properties of static analyses and their logics

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Abstract interpretation: computing on properties

```
readInt(x)
if x>0 :
   x:= pred(x)
x:= succ(x)
writeInt(x)
```

Q: Is output pos?

A: abstractly interpret domain Int by $Sign = \{neg, zero, pos, any\}$:

```
readSign(x)
if isPos(x):
   x:= pred♯(x)
x:= succ♯(x)
writeSign(x)
```

where

$$\begin{aligned} succ^\sharp(pos) &= pos \\ succ^\sharp(zero) &= pos \\ succ^\sharp(neg) &= any \\ succ^\sharp(any) &= any \end{aligned}$$

and

$$\begin{aligned} pred^\sharp(neg) &= neg \\ pred^\sharp(zero) &= neg \\ pred^\sharp(pos) &= any \\ pred^\sharp(any) &= any \end{aligned}$$

To answer the question, calculate the *static analysis*:

$$\{zero \mapsto pos,\ neg \mapsto any,\ pos \mapsto any,\ any \mapsto any\}$$

The Question is decided only for $zero$ — the static analysis is *sound* but *incomplete*.

Let $\mathit{Sign}' = \{\mathit{neg}, \leq 0, \mathit{zero}, \geq 0, \mathit{pos}, \mathit{any}\}$

```
readInt(x)
if x>0 :
   x:= pred(x)
x:= succ(x)
writeInt(x)
```

$\mathit{readSign}(x)$
if $\mathit{isPos}(x)$:
  x:= $\mathit{pred}^\sharp(x)$
x:= $\mathit{succ}^\sharp(x)$
$\mathit{writeSign}(x)$

where

$$\begin{aligned}
\mathit{succ}^\sharp(\mathit{pos}) &= \mathit{pos} \\
\mathit{succ}^\sharp(\geq 0) &= \mathit{pos} \quad (\text{-:} \\
\mathit{succ}^\sharp(\mathit{zero}) &= \mathit{pos} \\
\mathit{succ}^\sharp(\leq 0) &= \mathit{any} \\
\mathit{succ}^\sharp(\mathit{neg}) &= \leq 0 \quad (\text{-:} \\
\mathit{succ}^\sharp(\mathit{any}) &= \mathit{any}
\end{aligned}$$

and

$$\begin{aligned}
\mathit{pred}^\sharp(\mathit{neg}) &= \mathit{neg} \\
\mathit{pred}^\sharp(\leq 0) &= \mathit{neg} \quad (\text{-:} \\
\mathit{pred}^\sharp(\mathit{zero}) &= \mathit{neg} \\
\mathit{pred}^\sharp(\geq 0) &= \mathit{any} \\
\mathit{pred}^\sharp(\mathit{pos}) &= \geq 0 \quad (\text{-:} \\
\mathit{pred}^\sharp(\mathit{any}) &= \mathit{any}
\end{aligned}$$

The static analysis on Sign':

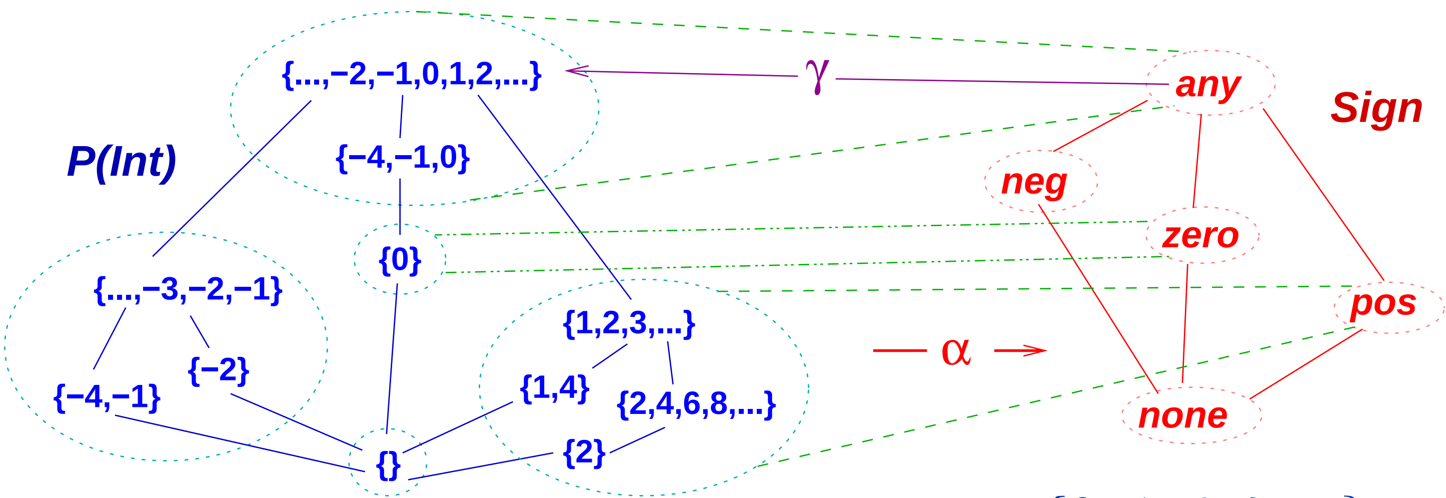
$\mathit{neg} \mapsto \mathit{neg}$ (-: $\quad \leq 0 \mapsto \mathit{any} \quad \mathit{zero} \mapsto \mathit{pos}$

$\mathit{pos} \mapsto \mathit{pos}$ (-: $\quad \mathit{any} \mapsto \mathit{any} \quad \geq 0 \mapsto \mathit{any}$)-:

Summary of the talk

1. Every static analysis employs an *abstract domain*, and every abstract domain possesses an *internal logic*.
2. Abstract state transformers must be *sound*, and perhaps they are *Backwards- Forwards-complete*.
3. Most program logics *extend* an internal logic, and their abstractions must be sound.
4. There are both *over-* and *underapproximating Galois connections* for approximating program logics; these define *F-, B-*, and *O-logical-completeness*.
5. The completeness notions are independent (and the independences are significant), but *coverings* are used to relate them.

Concrete data abstracts to (logical) properties



$\gamma(\mathrm{pos}) = \{1, 2, 3, \cdots\}$

$\gamma(\mathrm{zero}) = \{0\}$, etc.

$\gamma(\mathrm{any}) = \mathrm{Int}$, etc.

$\alpha\{2, 4, 6, 8, ...\} = \mathrm{pos}$

$\alpha\{-4, -1, 0\} = \mathrm{any}$

$\alpha\{0\} = \mathrm{zero}$

$\alpha\{\} = \mathrm{none}$, etc.

$(\mathcal{P}(\mathrm{Int}), \subseteq)\langle\alpha, \gamma\rangle(\mathrm{Sign}, \sqsubseteq)$ is a *Galois connection*: γ interprets the properties, and $\alpha(S) = \sqcap\{a \mid \gamma(a) \subseteq S\}$ maps concrete set S to the property that best describes it [CousotCousot77] .

We use such structures to do *static analysis*.

A Galois connection defines an **internal logic**

For $(PC, \subseteq)\langle\alpha, \gamma\rangle(A, \sqsubseteq)$, $S \in PC$, and $a \in A$, define

$$S \models a \text{ iff } S \subseteq \gamma(a) \text{ iff } \alpha(S) \sqsubseteq a$$

Example: For $Sign$, $\{2, 8\} \models pos$.

A Galois connection defines a logic with conjunction:

$$\phi \ ::= \ a \mid \phi_1 \sqcap \phi_2$$

because γ preserves $\sqcap$ as $\cap$ (that is, $\gamma(a_1 \sqcap a_2) = \gamma(a_1) \cap \gamma(a_2)$):

$$S \models a_1 \sqcap a_2 \text{ iff } S \models a_1 \text{ and } S \models a_2.$$

Example: In $Sign$, $\{2, 5\} \models pos \sqcap any$.

But the logic for $Sign$ excludes disjunction, e.g., $\{0\} \models any = neg \sqcup pos$, yet $\{0\} \not\models neg$ and $\{0\} \not\models pos$. This is because γ does not preserve $\sqcup$ as $\cup$.

Abstract transformers compute on properties

For $f : PC \rightarrow PC$, $f^\sharp : A \rightarrow A$ is *sound* iff

$$\alpha \circ f \sqsubseteq f^\sharp \circ \alpha \quad \text{iff} \quad f \circ \gamma \sqsubseteq \gamma \circ f^\sharp$$

α and γ act as *semi-homomorphisms*; $f^\sharp$ is a *postcondition transformer*.

Example: For $succ : \mathcal{P}(Int) \rightarrow \mathcal{P}(Int)$, $succ\{0\} = \{1\}$, $succ^\sharp(zero) = pos$.

Consequences: $f(S) \models f^\sharp(\alpha(S))$ and $f(\gamma(a)) \models f^\sharp(a)$.

For example, $\{0\} \models zero$ and $succ\{0\} \models succ^\sharp(zero) = pos$.

$f^\sharp_{best} = \alpha \circ f \circ \gamma$ is the best — strongest postcondition — transformer in A's internal logic.

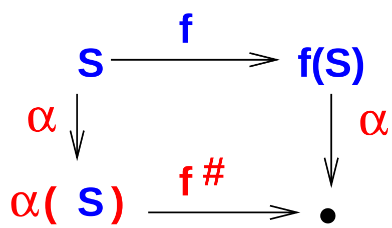
(Functional) completeness: *from semi-homomorphism to homomorphism*

For $f : PC \to PC$, $f^\sharp : A \to A$:

Backwards(α)-completeness

[Cousots79,GiacobazziJACM00] :

$$\alpha \circ f = f^\sharp \circ \alpha$$



α is a homomorphism from PC to A — it preserves f as $f^\sharp$.

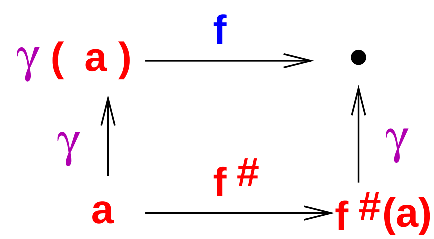
Corollary: $f^\sharp(\alpha(S)) \sqsubseteq a$ iff $f(S) \models a$.

That is, we can *decide* properties of f in A.

Forwards(γ)-completeness

[GiacobazziQuintarelli01] :

$$f \circ \gamma = \gamma \circ f^\sharp$$



γ is a homomorphism from A to PC — it preserves $f^\sharp$ as f.

Corollary: $S \models f^\sharp(a)$ iff $S \subseteq f(\gamma(a))$.

That is, $f^\sharp$ is a logical connective in A's internal logic (like $\sqcap$ is).

A typical program logic extends A's internal logic

Given Galois connection, $(\mathcal{P}(D), \subseteq)\langle\alpha, \gamma\rangle(A, \sqsubseteq)$, define $\mathcal{L}$ as follows:

$$a \in \mathrm{Prim} = A \text{ (the primitive assertions)}$$

$$\mathcal{L} \ni \phi ::= a \mid \phi_1 \wedge \phi_2 \mid \phi_1 \vee \phi_2 \mid [f]\phi$$

The interpretation, $[\![\cdot]\!] : \mathcal{L} \to \mathcal{P}(D)$, is defined as

$[\![a]\!] = \gamma(a)$

$[\![\phi_1 \wedge \phi_2]\!] = [\![\phi_1]\!] \cap [\![\phi_2]\!]$

$[\![\phi_1 \vee \phi_2]\!] = [\![\phi_1]\!] \cup [\![\phi_2]\!]$

$[\![[f]\phi]\!] = \widetilde{pre}_f[\![\phi]\!]$

where $\widetilde{pre}_f(S) = \{c \in D \mid f(c) \subseteq S\}$, and $f : D \to \mathcal{P}(D)$ is a state-transition function

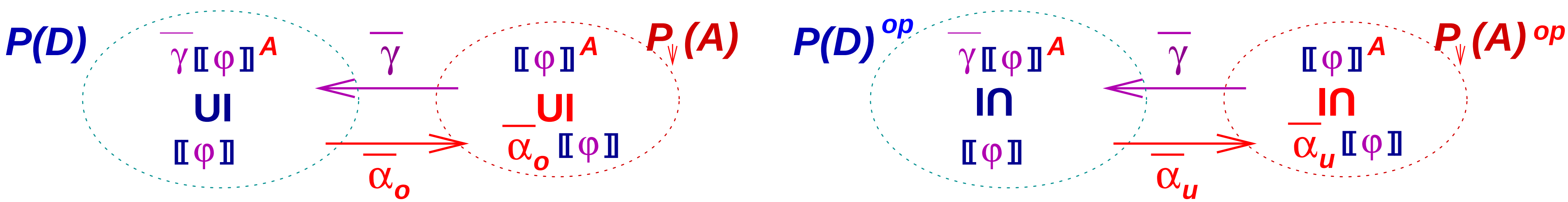
Say that $S \models \phi$ iff $S \subseteq [\![\phi]\!]$.

$\phi_1 \vee \phi_2$ and $[f]\phi$ might not fall in A's internal logic. (E.g., for Sign, there is no $\cup : \mathit{Sign} \times \mathit{Sign} \to \mathit{Sign}$ such that $\gamma(\mathit{neg} \cup \mathit{pos}) = [\![\phi_1 \vee \phi_2]\!]$.)

Q: How do we approximate $[\![\cdot]\!] : \mathcal{L} \to \mathcal{P}(D)$?
A: Define $[\![\cdot]\!]^A : \mathcal{L} \to \mathcal{P}_\downarrow(A)$

Given $(\mathcal{P}(D), \subseteq)\langle\alpha, \gamma\rangle(A, \sqsubseteq)$, we have *two* relevant Galois connections between $\mathcal{P}(D)$ and $\mathcal{P}_\downarrow(A)$:

Define $\overline{\gamma}(T) = \cup_{a \in T}\, \gamma(a)$.



Overapproximating abstraction:

$$\overline{\alpha_o}(S) = \bigcap\{T \mid S \subseteq \overline{\gamma}(T)\}$$
$$= \downarrow\{\alpha\{c\} \mid c \in S\}$$

where

$\downarrow T = \{a \mid$ exists $a' \in T, a \sqsubseteq a'\}$.

Underapproximating abstraction:

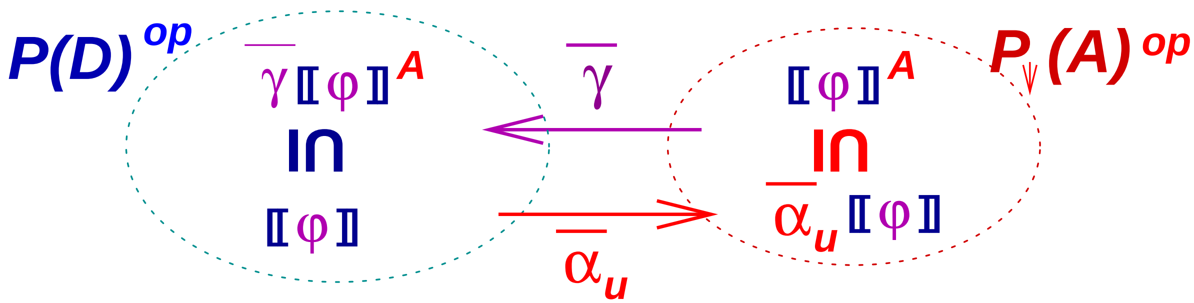
$$\overline{\alpha_u}(S) = \bigcup\{T \mid \overline{\gamma}(T) \subseteq S\}$$
$$= \{a \mid \gamma(a) \subseteq S\}$$

where

$(P, \sqsubseteq_P)^{op}$ is $(P, \sqsupseteq_P)$.

Abstracted, underapproximated logic

This is the best inductively defined underapproximation:



$$[\![a]\!]^A_{\text{best}} = \overline{\alpha_u}(\gamma(a))$$

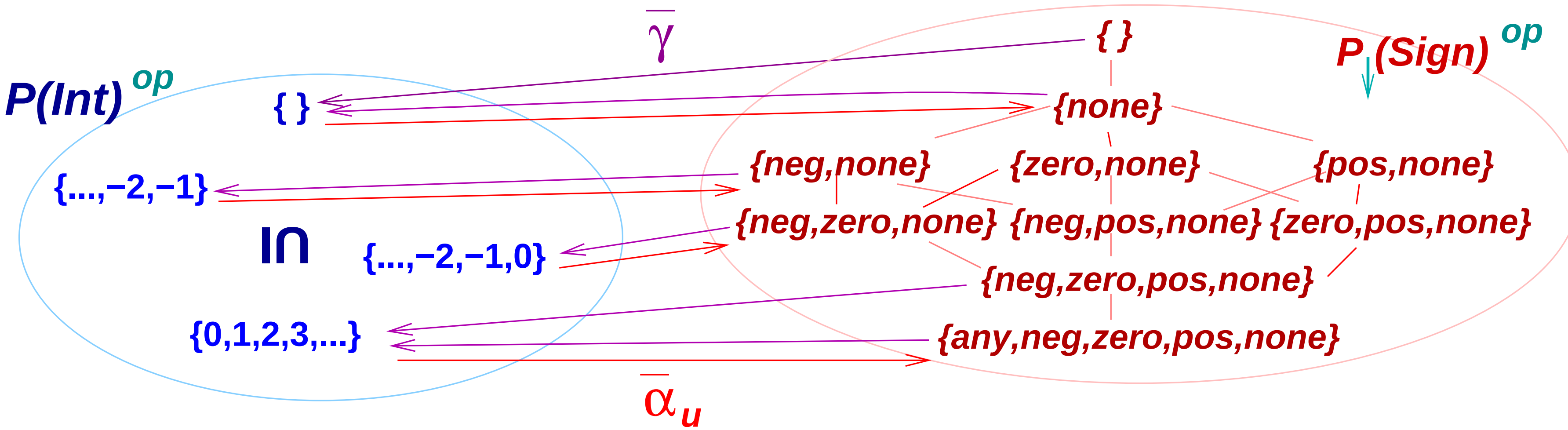
$$[\![\phi_1 \wedge \phi_2]\!]^A_{\text{best}} = [\![\phi_1]\!]^A_{\text{best}}(\overline{\alpha_u} \circ \cap \circ \overline{\gamma}^2)[\![\phi_2]\!]^A_{\text{best}}$$

$$[\![\phi_1 \vee \phi_2]\!]^A_{\text{best}} = [\![\phi_1]\!]^A_{\text{best}}(\overline{\alpha_u} \circ \cup \circ \overline{\gamma}^2)[\![\phi_2]\!]^A_{\text{best}}$$

$$[\![[f]\phi]\!]^A_{\text{best}} = (\overline{\alpha_u} \circ \widetilde{\text{pre}}_{f\sharp} \circ \overline{\gamma})[\![\phi]\!]^A_{\text{best}}$$

But it is not finitely computable in A....

Here is a less precise, but sound and finitely computable underapproximation for Sign:



$$[\![a]\!]^{\mathit{Sign}} = \ \downarrow\{a\} \ = \{a' \mid a' \sqsubseteq a\}$$

$$[\![\phi_1 \wedge \phi_2]\!]^{\mathit{Sign}} = [\![\phi_1]\!]^{\mathit{Sign}} \cap [\![\phi_2]\!]^{\mathit{Sign}}$$

$$[\![\phi_1 \vee \phi_2]\!]^{\mathit{Sign}} = [\![\phi_1]\!]^{\mathit{Sign}} \cup [\![\phi_2]\!]^{\mathit{Sign}}$$

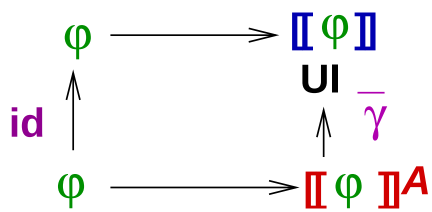
$$[\![[f]\phi]\!]^{\mathit{Sign}} = \ \widetilde{\mathit{pre}}_{f^\sharp}[\![\phi]\!]^{A}$$

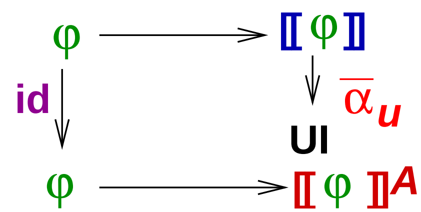
We have soundness: $\overline{\alpha_u}[\![\phi]\!] \supseteq [\![\phi]\!]^{\mathit{Sign}}_{\mathit{best}} \supseteq [\![\phi]\!]^{\mathit{Sign}}$, for all ϕ.

Logical soundness and completeness

$[\![\cdot]\!]^A : \mathcal{L} \to \mathcal{P}_\downarrow(\mathrm{Sign})$ is *sound* for $[\![\cdot]\!] : \mathcal{L} \to \mathcal{P}(D)$ iff

$$\overline{\gamma}[\![\phi]\!]^A \subseteq [\![\phi]\!] \quad \text{iff} \quad [\![\phi]\!]^A \subseteq \overline{\alpha_u}[\![\phi]\!]$$



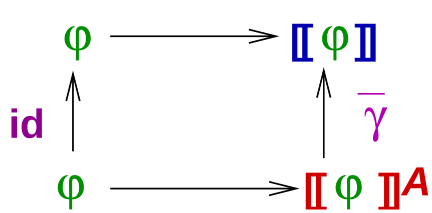


There are two forms of *completeness* of $[\![\cdot]\!]^A$ for $[\![\cdot]\!]$:

Forwards-completeness

[RanzatoTapparo06] :

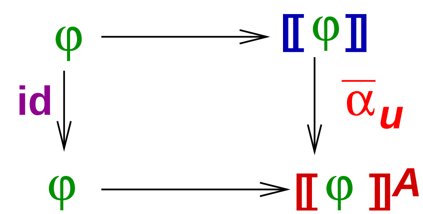
$$\overline{\gamma}[\![\phi]\!]^A = [\![\phi]\!]$$



Backwards-completeness

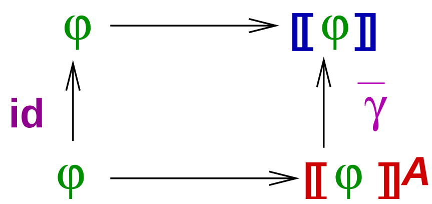
[CousotCousot00] :

$$[\![\phi]\!]^A = \overline{\alpha_u}[\![\phi]\!]$$

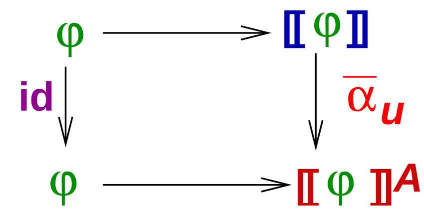


Strong, best, and lower preservation

F-complete: $\overline{\gamma}[\![\phi]\!]^A = [\![\phi]\!]$



B-complete: $[\![\phi]\!]^A = \overline{\alpha_u}[\![\phi]\!]$



- *best preservation:* for all $\phi \in \mathcal{L}$ and $T \in \mathcal{P}_{\downarrow}(A)$,
  $T \subseteq [\![\phi]\!]^A$ iff $\overline{\gamma}(T) \subseteq [\![\phi]\!]$.

- *strong preservation:* for all $\phi \in \mathcal{L}$ and $S \in \mathcal{P}(D)$,
  $S \subseteq [\![\phi]\!]$ iff $\overline{\alpha_o}(S) \subseteq [\![\phi]\!]^A$.

- *lower preservation:* for all $\phi \in \mathcal{L}$ and $S \in \mathcal{P}(D)$,
  $[\![\phi]\!] \subseteq S$ iff $[\![\phi]\!]^A \subseteq \overline{\alpha_u}(S)$.

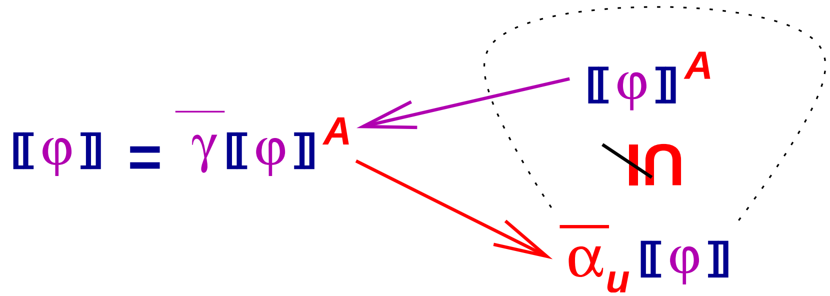
Theorem:

- B-complete iff best preservation

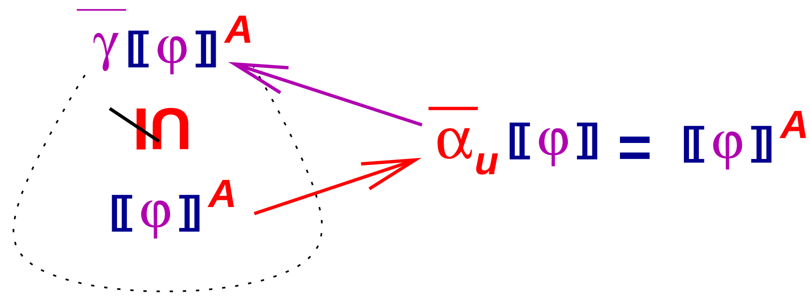
- F-complete iff strong preservation iff lower preservation

The two forms of completeness are **independent**

F-complete and not B-complete:



B-complete and not F-complete:



Absence of B-completeness: we fail to validate $any \in [\![neg \vee zero \vee pos]\!]^{Sign}$. We must use a *focus* operation: $focus(any) = \{neg, zero, pos\}$ and validate $a \in [\![neg \vee zero \vee pos]\!]^{Sign}$, for all $a \in focus(any)$ [Dams04,Sagiv02] .

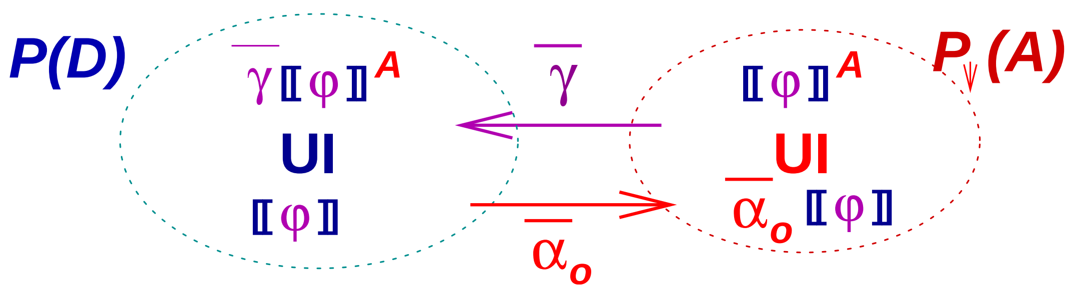
Absence of F-completeness: Say that $eq1 \in \mathcal{L}$ and $[\![eq1]\!] = \{1\}$, making $[\![eq1]\!]^{Sign} = \{none\}$. Then, a static analysis of

```
x:= 1; if x=1 then safe() else error()
```

announces `error()` is reachable. *Counterexample guided abstraction refinement (CEGAR)* repairs the problem by adding new elements to $Sign$ [Ball02,Clarke00,Saidi00] .

O-completeness: *subset-inclusion completeness*

For $\mathcal{P}(\mathrm{D})\langle\overline{\alpha_{\mathrm{o}}}, \overline{\gamma}\rangle\mathcal{P}_{\downarrow}(A)$,



the inclusion,

$$\overline{\alpha_{\mathrm{o}}}[\![\phi]\!] \subseteq [\![\phi]\!]^{A}$$

does not ensure soundness. Nonetheless, we can define one more variant of completeness (which *is* sound):

$$[\![\cdot]\!]^{A} \text{ is } B(\overline{\alpha_{\mathrm{o}}})\text{-complete for } [\![\cdot]\!] \text{ iff } \overline{\alpha_{\mathrm{o}}}[\![\phi]\!] = [\![\phi]\!]^{A}.$$

We use *O-complete* as a synonym for *B($\overline{\alpha_{\mathrm{o}}}$)-complete*.

With coverings, *we make many connections*

For $[\![\cdot]\!] : \mathcal{L} \to \mathcal{P}(\mathrm{D})$ and $\overline{\gamma} : \mathrm{Q} \to \mathcal{P}(\mathrm{D})$,

$\overline{\gamma}$ *covers* $[\![\cdot]\!]$ iff for all $\phi \in \mathcal{L}$, $[\![\phi]\!] \in \overline{\gamma}[\mathrm{Q}]$.

For $[\![\cdot]\!]^A : \mathcal{L} \to \mathcal{P}_\downarrow(A)$ and $\overline{\alpha} : \mathrm{P} \to \mathcal{P}_\downarrow(A)$,

$\overline{\alpha}$ *covers* $[\![\cdot]\!]^A$ iff for all $\phi \in \mathcal{L}$, $[\![\phi]\!]^A \in \overline{\alpha}[\mathrm{P}]$.

Theorem:

- If $[\![\cdot]\!]^A$ is F-complete for $[\![\cdot]\!]$ and $\overline{\alpha_u}$ covers $[\![\cdot]\!]^A$, then $[\![\cdot]\!]^A$ is B-complete.
- If $[\![\cdot]\!]^A$ is B-complete for $[\![\cdot]\!]$ and $\overline{\gamma}$ covers $[\![\cdot]\!]$, then $[\![\cdot]\!]^A$ is F-complete.
- If $[\![\cdot]\!]^A$ is F-complete for $[\![\cdot]\!]$ and $\overline{\alpha_o}$ covers $[\![\cdot]\!]^A$, then $[\![\cdot]\!]^A$ is O-complete.
- If $[\![\cdot]\!]^A$ is O-complete for $[\![\cdot]\!]$ and $\overline{\gamma}$ covers $[\![\cdot]\!]$, then $[\![\cdot]\!]^A$ is *sound* as well as F-complete.

An application: partition domains

Let D and A be discretely ordered sets, and let $\delta : D \to A$ be an onto function, defining the *partition*, $c \sim_\delta c'$ iff $\delta(c) = \delta(c')$. Define $\gamma : A \to \mathcal{P}(D)$ as $\gamma(a) = \delta^{-1}(a)$. We have this propositional logic:

$$[\![a]\!] = \gamma(a) \qquad [\![\phi_1 \wedge \phi_2]\!] = [\![\phi_1]\!] \cap [\![\phi_2]\!]$$

$$[\![\neg\phi]\!] = \sim[\![\phi]\!] \qquad [\![\phi_1 \vee \phi_2]\!] = [\![\phi_1]\!] \cup [\![\phi_2]\!]$$

The abstract logic,

$$[\![a]\!]^A = \overline{\alpha_u}(\gamma(a)) \qquad [\![\phi_1 \wedge \phi_2]\!]^A = [\![\phi_1]\!]^A \cap [\![\phi_2]\!]^A$$

$$[\![\neg\phi]\!]^A = \sim [\![\phi]\!]^A \qquad [\![\phi_1 \vee \phi_2]\!]^A = [\![\phi_1]\!]^A \cup [\![\phi_2]\!]^A$$

is F-complete and equals $[\![\cdot]\!]^A_{best}$. Since both $\overline{\alpha_u}$ and $\overline{\alpha_o}$ cover $[\![\cdot]\!]^A$, the logic is also B- and O-complete.

The usual application of a partition domain is to model checking, whose logic includes the modality, $[f]\phi$, for $f : D \rightarrow \mathcal{P}(D)$, which is abstracted by a sound $f^\sharp : A \rightarrow \mathcal{P}(A)$ as follows:

$$[\![[f]\phi]\!]^A = \widetilde{pre}_{f^\sharp_{best}} [\![\phi]\!]^A, \quad \text{where } \widetilde{pre}_{f^\sharp}(T) = \{a' \mid f^\sharp(a') \subseteq T\}.$$

We know that $\widetilde{pre}_{f^\sharp_{best}} \; = \; (\widetilde{pre}_f)^\sharp{}_{best} \; = \; \overline{\alpha_u} \circ \widetilde{pre}_f \circ \overline{\gamma}$ [Schmidt06] .
The definition is sound but might not be complete — *this depends on* f:

Theorem: For $\widetilde{pre}_f : \mathcal{P}(D) \rightarrow \mathcal{P}(D)$, $f : D \rightarrow \mathcal{P}(D)$, and $f^* : \mathcal{P}(D) \rightarrow \mathcal{P}(D)$, defined as $f^*(S) = \cup_{c \in S} f(c)$,

1. $\widetilde{pre}_f$ is F($\overline{\gamma}$)-complete iff f^* is B($\overline{\alpha_o}$)-complete.

2. $\widetilde{pre}_f$ is B($\overline{\alpha_u}$)-complete iff f^* is F($\overline{\gamma}$)-complete.

Summary

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2. Abstract state transformers must be *sound*, and perhaps they are *Backwards- Forwards-complete*.

3. Most program logics *extend* an internal logic, and their abstractions must be sound.

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