Claim 1: Let $G = (V, T, P, S)$ be a CFG, and suppose $\beta_1 \Rightarrow^* \beta_2$ for some $\beta_1, \beta_2 \in (V \cup T)^*$. Then for every $\alpha, \gamma \in (V \cup T)^*$, $\alpha \beta_1 \gamma \Rightarrow \alpha \beta_2 \gamma$.

**Proof:** By induction on $\Rightarrow^*$.

**Base:** $\beta_1 = \beta_2$. By reflexivity, $\alpha \beta_1 \gamma \Rightarrow \alpha \beta_2 \gamma$.

**IH:** Assume that for some $\beta_1 \Rightarrow^* \beta_2$, $\alpha \beta_1 \gamma \Rightarrow^* \alpha \beta_2 \gamma$ for all $\alpha, \gamma \in (V \cup T)^*$.

**IS:** Suppose $\beta_2 \Rightarrow \beta_3$. Then $\alpha \beta_2 \gamma \Rightarrow \alpha \beta_3 \gamma$ for $\alpha, \gamma \in (V \cup T)^*$. Then $\alpha \beta_1 \gamma \Rightarrow \alpha \beta_3 \gamma$.

Claim 2: Let $G = (V, T, P, S)$ be a CFG, and let $\alpha_1$ and $\beta_1$ be strings in $(V \cup T)^*$. Then $\alpha_1 \beta_1 \Rightarrow \gamma \in (V \cup T)^*$ iff $\gamma = \alpha_2 \beta_2$ such that $\alpha_1 \Rightarrow^* \alpha_2$ and $\beta_1 \Rightarrow^* \beta_2$.

$\Rightarrow$: By induction on $\Rightarrow^*$.

**Base:** $\gamma = \alpha_1 \beta_1$. By reflexivity, $\alpha_1 \Rightarrow \alpha_1$ and $\beta_1 \Rightarrow \beta_1$.

**IH:** Assume that for some $\alpha_1 \beta_1 \Rightarrow \gamma$, $\gamma = \alpha_2 \beta_2$ such that $\alpha_1 \Rightarrow^* \alpha_2$ and $\beta_1 \Rightarrow^* \beta_2$. 

1

2
IS: Suppose $\gamma \Rightarrow \gamma_1$.

- Then $\gamma = \alpha_3 A\beta_3$ and $\gamma_1 = \alpha_3 \gamma_2 \beta_3$ such that $A \rightarrow \gamma_2 \in P$.

- By the IH, $\alpha_3 A\beta_3 = \gamma = \alpha_2 \beta_2$ such that $\alpha_1 \Rightarrow \alpha_2$ and $\beta_1 \Rightarrow \beta_2$.

- W.o.l.o.g., suppose $\alpha_2 = \alpha_3 A\gamma_3$ and $\beta_3 = \gamma_3 \beta_2$. (The case in which $A$ is in $\beta_2$ is handled similarly.)

- Then $\gamma_1 = \alpha_3 \gamma_2 \gamma_3 \beta_2$ such that $\alpha_1 \Rightarrow \alpha_3 \gamma_2 \gamma_3$ and $\beta_1 \Rightarrow \beta_2$.

$\Leftarrow$: Suppose $\alpha_1 \Rightarrow \alpha_2$ and $\beta_1 \Rightarrow \beta_2$. Then from Claim 1:

$$\alpha_1 \beta_1 \Rightarrow \alpha_2 \beta_1 \Rightarrow \alpha_2 \beta_2.$$
Let $L = \{xx^Rw \mid x, w \in \{0, 1\}^*, x \neq \epsilon\}$. Let $G = (\{S, A, B\}, \{0, 1\}, P, S)$ where $P$ is given by

\[
S \rightarrow AB \\
A \rightarrow 0A0 \mid 1A1 \mid 00 \mid 11 \\
B \rightarrow 0B \mid 1B \mid \epsilon
\]

**Lemma 1:** For every $w \in \{0, 1\}^*$, $B \Rightarrow^* w$.

**Lemma 2:** For $w \in \{0, 1\}^*$, $A \Rightarrow^* w$ iff $w \in \{xx^R \mid x \in \{0, 1\}^+\}$.

**Claim:** $L(G) = L$.

**Proof:**

$w \in L(G)$ $\iff$ $S \Rightarrow^* w$

$\iff AB \Rightarrow^* w$

$\iff w = uv, A \Rightarrow^* u, B \Rightarrow^* v$

$\iff w = uv, u \in \{xx^R \mid x \in \{0, 1\}^+\}, v \in \{0, 1\}^*$

$\iff w \in L$.
Well-Founded Relations

Defn: Let $A$ be a set, and $R \subseteq A \times A$. $R$ is said to be well-founded if every nonempty subset $B$ of $A$ has a least element with respect to $R$; i.e., if $\emptyset \neq B \subseteq A$, then there is an $x \in B$ such that for every pair $\langle x, y \rangle \in R$, $y \notin B$.

Examples:

- $>$ on $\mathbb{N}$.
- $\{ \langle w, y \rangle \in \Sigma^* \times \Sigma^* \mid w = yz, z \neq \epsilon \}$.
- $\{ \langle w, y \rangle \in \Sigma^* \times \Sigma^* \mid w = xyz, xz \neq \epsilon \}$.

Principle of Induction on a Well-Founded Relation: Let $A$ be a set and $R \subseteq A \times A$ be well-founded. Suppose $B \subseteq A$ such that for every $x \in A$,

$$(\forall y \in A : \langle x, y \rangle \in R \Rightarrow y \in B) \Rightarrow x \in B.$$  

Then $B = A$.

Proof: By contradiction. Suppose $A - B \neq \emptyset$. Because $R$ is well-founded, $A - B$ has a least element $x$ w.r.t. $R$. Then $\forall y \in A : \langle x, y \rangle \in R \Rightarrow y \in B$. Therefore, $x \in B$ — a contradiction.
**Example:** Let \( L = \{w \in \{0, 1\}^* \mid w = w^R\} \).
Let \( G = (\{S\}, \{0, 1\}, P, S) \), where \( P \) is given by
\[
S \to 0S0 \mid 1S1 \mid 0 \mid 1 \mid \epsilon
\]

**Claim:** \( L = L(G) \).

\( \subseteq \): By induction on the proper substring relation on \( \{0, 1\}^* \).

**Base:** \( w \in \{\epsilon, 0, 1\} \). Then \( S \Rightarrow \epsilon = w \), so \( w \in L(G) \).

**IH:** Assume for some \( w \in L - \{\epsilon, 0, 1\} \), that for every proper substring \( x \) of \( w \), if \( x \in L \), then \( x \in L(G) \).

**IS:** Suppose w.o.l.o.g. that \( w = 0y \), \( y \neq \epsilon \) (the other case is symmetric).

- Because \( w^R = w = 0y \), \( w = y^R0 \); hence, \( w = 0x0 \) for some \( x \in \{0, 1\}^* \).
- Because \( w = w^R \), \( 0x0 = 0x^R0 \); i.e., \( x = x^R \), so \( x \in L \).
- By the IH, \( x \in L(G) \), so \( S \Rightarrow^* x \).
- \( S \Rightarrow 0S0 \Rightarrow^* 0x0 = w \), so \( w \in L(G) \).

\( \supseteq \): We can show by induction on \( \Rightarrow^* \) that if \( S \Rightarrow^* \alpha \), then \( \alpha = \alpha^R \) and \( \alpha \) contains at most one \( S \).
Ambiguous CFGs

Let \( G = (\{S\}, T, P, S) \), where

\[ T = \{\text{if, then, else, } b, c\} \]

and \( P \) is given by

\[ S \rightarrow \text{if } b \text{ then } S \mid \text{if } b \text{ then } S \text{ else } S \mid c \]

Let \( w = \text{if } b \text{ then if } b \text{ then } c \text{ else } c \).

We can parse \( w \) using \( G \) as follows:

```
S
  /\  /
if b then S
  /\  /
if b then S else S
     /
    c
    /
    c
```
We can also parse $w$ as follows:

$$
S
\begin{array}{c}
\text{if } b \text{ then } S \text{ else } S \\
\text{if } b \text{ then } S \ c \\
\ c
\end{array}
$$

**Defn:** A CFG $G = (V, T, P, S)$ is said to be *ambiguous* if there are strings $w \in T^*$, $\alpha$, $\beta$, $\gamma_1$, $\gamma_2 \in (V \cup T)^*$, and a symbol $A \in V$, such that

- $S \Rightarrow^* \alpha A \beta \Rightarrow^* \alpha \gamma_1 \beta \Rightarrow^* w$;
- $S \Rightarrow^* \alpha A \beta \Rightarrow^* \alpha \gamma_2 \beta \Rightarrow^* w$; and
- $\gamma_1 \neq \gamma_2$.

**Defn:** A CFL $L$ is said to be *inherently ambiguous* if there there is no unambiguous CFG $G$ such that $L(G) = L$. 

13