**Defn:** Let $L_{eq}$ be the set of all strings over \{0, 1\} having an equal number 0s and 1s.

**Question:** Is $L_{eq}$ regular?

**Theorem 4.1: (The pumping lemma for regular languages)** Let $L$ be a regular language. Then there is an $n \in \mathbb{N}$ such that for every $w \in L$ for which $|w| \geq n$, we can write $w = xyz$ such that:

1. $y \neq \epsilon$;
2. $|xy| \leq n$; and
3. $\forall k \geq 0 : xy^kz \in L$.

**Claim:** $L_{eq}$ is not regular.

**Proof:** By contradiction. Assume $L_{eq}$ is regular. Let $n$ be the constant guaranteed by the pumping lemma. Let $w = 0^n1^n$. Clearly, $w \in L_{eq}$. By the pumping lemma, we can write $w = xyz$ such that $y \neq \epsilon$, $|xy| \leq n$, and $\forall k \in \mathbb{N} : xy^kz \in L_{eq}$. Because $|xy| \leq n$, $y = 0^i$ such that $0 < i \leq n$. Then $xy^0z = 0^{n-i}1^n \not\in L_{eq}$ — a contradiction.
Proof of Pumping Lemma (sketch):

- Let $A = (Q, \Sigma, \delta, q_0, F)$ be a DFA s.t. $L(A) = L$.
- Let $n = |Q|$, $w \in L$, $|w| \geq n$.
- For $0 \leq i \leq n$, let $w_i$ denote the prefix of $w$ with length $i$.
- By the Pigeonhole Principle, 
  $\exists 0 \leq i < j \leq n : \hat{\delta}(q_0, w_i) = \hat{\delta}(q_0, w_j)$.
- Let $x = w_i, xy = w_j, xyz = w$ (note that $|xy| \leq n, y \neq \epsilon$).

Proof continued:

- By induction on $k$, $\hat{\delta}(q_0, xy^k) = \hat{\delta}(q_0x)$ for all $k \in \mathbb{N}$.

$$\begin{align*}
\hat{\delta}(q_0, xy^k z) &= \hat{\delta}(\hat{\delta}(q_0, xy^k), z) \\
&= \hat{\delta}(\hat{\delta}(q_0, x), z) \\
&= \hat{\delta}(\hat{\delta}(q_0, xy), z) \\
&= \hat{\delta}(q_0, xyz) \\
&\in F.
\end{align*}$$

- Therefore, $xy^k z \in L(A) = L$. 
**Theorem 4.5:** The set of regular languages over $\Sigma$ is closed under complementation; i.e., if $L$ is a regular language over $\Sigma$, then $\Sigma^* - L$ is regular.

**Proof:** Let $A = (Q, \Sigma, \delta, q_0, F)$ be a DFA s.t. $L(A) = L$. Let $A' = (Q, \Sigma, \delta, q_0, Q - F)$. Then

$$\Sigma^* - L = \Sigma^* - \{w \in \Sigma^* \mid \hat{\delta}(q_0, w) \in F\}$$

$$= \{w \in \Sigma^* \mid \hat{\delta}(q_0, w) \in Q - F\}$$

$$= L(A').$$

Thus, $\Sigma^* - L$ is regular.

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**Theorem 4.8:** The set of regular languages over $\Sigma$ is closed under intersection.

**Proof:** Let $L_1$ and $L_2$ be regular languages over $\Sigma$. By DeMorgan’s laws,

$$L_1 \cap L_2 = \overline{L_1} \cup \overline{L_2}.$$

Because the set of regular languages over $\Sigma$ is closed under union and complementation, $L_1 \cap L_2$ is regular.
**Defn:** Let $\Sigma$ and $\Delta$ be two alphabets. A *homomorphism* from $\Sigma^*$ to $\Delta^*$ is any function $h : \Sigma^* \to \Delta^*$ such that $\forall x, y \in \Sigma^*$, $h(xy) = h(x)h(y)$.

**Defn:** Let $\Sigma$ and $\Delta$ be two alphabets, and let $R \subseteq \Sigma^* \times \Delta^*$. We define $\tilde{R} : 2^{\Sigma^*} \to 2^{\Delta^*}$ such that

$$\tilde{R}(L) = \{ y \in \Delta^* | \exists x \in L : \langle x, y \rangle \in R \}$$

**Example:** Let $h : \{a, b\}^* \to \{0, 1\}^*$ be the homomorphism defined by

$$h(a) = 01$$
$$h(b) = 10.$$

Then $\tilde{h}(a^*b^*) = (01)^*(10)^*$. 
For a string \( w \in \{a, b\}^* \), let \( \overline{w} \) denote the homomorphism defined by
\[
\overline{a} = b \\
\overline{b} = a.
\]

Let \( L = \{xx^Rw \mid x, w \in \{0, 1\}^*, x \neq \epsilon\} \). Then
\[
\overline{h^{-1}}(L) = \{x \in \{a, b\}^* \mid \exists y \in L : h(x) = y\} \\
= \{xx^Rw \in \{a, b\}^* \mid x \neq \epsilon\}
\]

**Theorem 4.14:** Let \( h : \Sigma^* \rightarrow \Delta^* \) be a homomorphism. If \( L \subseteq \Sigma^* \) is regular, then \( \overline{h}(L) \) is also regular.

**Theorem 4.16:** If \( h : \Sigma^* \rightarrow \Delta^* \) is a homomorphism and \( L \subseteq \Delta^* \) is regular, then \( \overline{h^{-1}}(L) \) is regular.

**Claim:** The language \( L = \{xx^Rw \mid x, w \in \{0, 1\}^*, x \neq \epsilon\} \) is not regular.

**Proof idea:**
\[
\overline{h^{-1}}(L) \cap a^*b^* = \{a^ib^j \mid 0 < i \leq j\}.
\]
Proof of Theorem 4.14 (sketch): By induction on $L \in R(\Sigma)$.

Base Case 1: $\tilde{h}(\emptyset) = \emptyset$.

Base Case 2: $\tilde{h}$$$\{\varepsilon\} = \{\varepsilon\}$.

Base Case 3: For $a \in \Sigma$, $\tilde{h}(\{a\}) = \{h(a)\}$.

IH: Assume that for $L_1, L_2 \in R(\Sigma)$, $\tilde{h}(L_1)$ and $\tilde{h}(L_2)$ are regular.

IS 1: $\tilde{h}(L_1 \cup L_2) = \tilde{h}(L_1) \cup \tilde{h}(L_2)$.

IS 2: $\tilde{h}(L_1L_2) = \tilde{h}(L_1)\tilde{h}(L_2)$.

IS 3: $\tilde{h}(L_1^*) = \tilde{h}(L_1)^*$.

Proof of Theorem 4.16 (sketch):

- Let $A = (Q, \Delta, \delta_A, q_0, F)$ be a DFA, $L(A) = L$.

- Let $B = (Q, \Sigma, \delta_B, q_0, F)$, where $\delta_B(q, a) = \hat{\delta}(q, h(a))$.

- By right induction on $w \in \Sigma^*$, $\hat{\delta}_B(q_0, w) = \hat{\delta}_A(q_0, h(w))$.

- Therefore, $\hat{h}^{-1}(L) = L(B)$. 