\(\epsilon\)-NFAs

**Defn:** An \(\epsilon\)-NFA is a 5-tuple \((Q, \Sigma, \delta, q_0, F)\), where \(Q\), \(\Sigma\), \(q_0\), and \(F\) are as defined for DFAs, and

\[
\delta : Q \times (\Sigma \cup \{\epsilon\}) \rightarrow 2^Q.
\]

\(\epsilon\)-closure

**Defn:** Let \(A = (Q, \Sigma, \delta, q_0, F)\) be an \(\epsilon\)-NFA. For \(q \in Q\), we define the \(\epsilon\)-closure of \(q\) to be the least set \(S\) such that

- \(q \in S\) and
- if \(p \in S\) and \(r \in \delta(p, \epsilon)\), then \(r \in S\).

We denote the \(\epsilon\)-closure of \(q\) by \text{ECLOSE}(q).
**Defn:** Let $A = (Q, \Sigma, \delta, q_0, F)$ be an $\epsilon$-NFA.

We define $\hat{\delta} : Q \times \Sigma^* \rightarrow 2^Q$ as follows:

- $\hat{\delta}(q, \epsilon) = \text{ECLOSE}(q)$ for all $q \in Q$; and
- $\hat{\delta}(q, ay) = \bigcup_{p \in \text{ECLOSE}(q)} \bigcup_{r \in \delta(p, a)} \hat{\delta}(r, y)$ for all $q \in Q$, $a \in \Sigma$, $y \in \Sigma^*$.

**Defn:** Let $A = (Q, \Sigma, \delta, q_0, F)$ be an $\epsilon$-NFA.

We define the *language accepted by* $A$ to be

$$L(A) = \{x \in \Sigma^* | \hat{\delta}(q_0, x) \cap F \neq \emptyset\}.$$ 

**Claim:** Let $A = (Q, \Sigma, \delta, q_0, F)$ be an $\epsilon$-NFA.

Then for all $q \in Q$, $a \in \Sigma$, and $y \in \Sigma^*$,

$$\hat{\delta}(q, ya) = \bigcup_{q \in \hat{\delta}(q, y)} \left( \bigcup_{r \in \delta(p, a)} \text{ECLOSE}(r) \right).$$

**Theorem 2.22:** A language $L$ is regular iff there is some $\epsilon$-NFA $N$ such that $L(N) = L$. 

3
Operations on Languages

**Defn:** Let $L_1$ and $L_2$ be two languages. The *concatenation* of $L_1$ and $L_2$ is defined to be

$$L_1L_2 = \{ xy \mid x \in L_1, y \in L_2 \}.$$  

**Defn:** Let $L$ be a language. We define the *Kleene closure of $L$*, denoted $L^*$, to be the least set $S$ such that

- $\epsilon \in S$; and
- if $x \in L$ and $y \in S$, then $xy \in S$.

**Defn:** Let $L \subseteq \Sigma^*$. We recursively define:

- $L^0 = \{ \epsilon \}$; and
- $L^{i+1} = LL^i$ for $i \in \mathbb{N}$.

**Defn:** Let $L \subseteq \Sigma^*$. Then

$$L^+ = \bigcup_{i>0} L^i.$$  

**Claim:** Let $L \subseteq \Sigma^*$. Then

$$L^* = \bigcup_{i \in \mathbb{N}} L^i.$$
Proof sketch:

⊆: By induction on $y \in L^*$. 

**Base:** $y = \epsilon$. Then $y \in L^0$.

**IH:** Assume that for some $y \in L^*$, $y \in \bigcup_{i \in \mathbb{N}} L^i$.

**IS:** Let $x \in L$.

- For some $i$, $y \in L^i$.
- Then $xy \in L^{i+1}$.

⊇: We show by induction on $i \in \mathbb{N}$ that $L^i \subseteq L^*$.

**Base:** $i = 0$. Then 

$$L^0 = \{\epsilon\}$$

$$\subseteq L^*.$$ 

**IH:** Assume that for some $i \in \mathbb{N}$, $L^i \subseteq L^*$.

**IS:**

$$L^{i+1} = LL^i$$

$$\subseteq LL^*$$

$$\subseteq L^*.$$
**Defn:** Let \( \Sigma \) be an alphabet. We define \( R(\Sigma) \) to be the least set \( R \) such that

- \( \emptyset \in R; \)
- \( \{\epsilon\} \in R; \)
- for each \( a \in \Sigma, \{a\} \in R; \)
- if \( L_1 \in R \) and \( L_2 \in R \), then \( L_1 \cup L_2 \in R; \)
- if \( L_1 \in R \) and \( L_2 \in R \), then \( L_1L_2 \in R; \) and
- if \( L \in R \), then \( L^* \in R. \)

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**Regular Expressions**

**Defn:** A *regular expression* is a description of a language in \( R(\Sigma) \) using the following notation:

- \( \epsilon \) denotes \( \{\epsilon\}; \)
- \( a \) denotes \( \{a\} \) for \( a \in \Sigma; \) and
- union is denoted by \( + \).
Precedence in regular expressions:

1. Kleene closure
2. Concatenation
3. Union

**Theorem 3.4:** Let $L$ be a regular language over $\Sigma$. Then $L \in R(\Sigma)$.

**Lemma 1:** The set of regular languages over $\Sigma$ is closed under union.

**Proof sketch:**

![Diagram of two automata](image)

$A_1$ and $A_2$ are automata for languages $L_1$ and $L_2$, respectively. The union $L_1 \cup L_2$ is represented by the automaton for $L$.
Lemma 2: The set of regular languages over $\Sigma$ is closed under concatenation.

Proof sketch:

Lemma 3: The set of regular languages over $\Sigma$ is closed under Kleene closure.

Proof sketch:
**Theorem 3.7:** If $L \in R(\Sigma)$, then $L$ is regular.

**Proof sketch:** By induction on $L \in R(\Sigma)$.

**Base:** $\emptyset$, $\{\epsilon\}$, and $\{a\}$ for $a \in \Sigma$ are easily seen to be regular.

**IH:** Assume that some $L_1, L_2 \in R(\Sigma)$ are regular.

**IS:** By Lemmas 1-3, $L_1 \cup L_2$, $L_1L_2$, and $L_1^*$ are regular.