Independent Set (IS)

**Input:** An undirected graph $G$ and a natural number $k$.

**Question:** Does $G$ contain and *independent set* of size $k$? I.e., is there a set of $k$ nodes of $G$ in which no two nodes are adjacent?

**Claim:** $\text{IS} \in \mathcal{NP}$.
- Guess a set of nodes.
- Count this set.
- Verify that it is an independent set.

**Theorem 10.18:** IS is $\mathcal{NP}$-complete.

**Proof sketch:** We will show that $3\text{SAT} \leq_p \text{IS}$.
- Let $\mathcal{F}$ be a 3CNF formula.
- We will construct a graph $G$ and a positive integer $k$ such that $G$ has an independent set iff $\mathcal{F}$ is satisfiable.
General Strategy for Finding Reductions

- Focus on what the nondeterministic TMs guess.
- Try to find a correspondence between what is guessed; e.g., between a satisfying assignment and an independent set.

Construction of $G$, Step 1

For each variable $v_i$:

$v_i \rightarrow \neg v_i$
Construction of $G$, Step 2

For each conjunct $c_j$:

For $c_j = \alpha_{j1} \lor \alpha_{j2} \lor \alpha_{j3}$:
Example: \((v_1 \lor \neg v_3 \lor \neg v_4) \land (\neg v_1 \lor v_2 \lor \neg v_4)\)

- \(G\) contains \(2n + 3m\) nodes, where \(n\) is the number of variables and \(m\) is the number of conjuncts.

- Let \(k = n + m\).

- For any satisfying assignment, the set of false literals and one node from each clause adjacent to a true literal forms an independent set of size \(k\).

- For any independent set of size \(k\), we can form a satisfying assignment by setting literals in the independent set to false.
Node Cover (NC)

**Input:** A graph $G$ and a natural number $k$.

**Question:** Does $G$ have a *node cover* of size at most $k$?
I.e., is there a set of nodes such that every edge is incident on some node in that set?

**Claim:** Let $N$ be the set of nodes in a graph $G$ and $C \subseteq N$. Then $C$ is a node cover iff $N - C$ is an independent set.

**Theorem 10.20:** NC is $\mathcal{NP}$-complete.