Claim: Let $L$ be an RE language such that $\overline{L}$ is also RE. Then $L$ is recursive.

Proof sketch:

- Let $L(M) = L$.
- Let $L(M') = \overline{L}$.
- We construct $M''$ to simulate $M$ and $M'$ in parallel using two tapes.
- Because either $M$ or $M'$ must accept $x$, $M''$ halts on all inputs.

Defn: A language $L$ is said to be co-RE if $\overline{L}$ is RE.

Corollary: If $L$ is a recursive language, then $L$ is co-RE.
Claim: For any language $L$, $L \leq L$.

Claim: Let $L_1$, $L_2$, and $L_3$ be languages such that $L_1 \leq L_2$ and $L_2 \leq L_3$. Then $L_1 \leq L_3$.

Defn: $L_1$ and $L_2$ are said to be many-one equivalent ($L_1 \equiv L_2$) if $L_1 \leq L_2$ and $L_2 \leq L_1$.

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Rice’s Theorem

Defn: Let $RE(\Sigma)$ denote the set of RE languages over $\Sigma$. A property of $RE(\Sigma)$ is any subset of $RE(\Sigma)$. $\emptyset$ and $RE(\Sigma)$ are trivial properties; all other properties are nontrivial.

Defn: Let $\mathcal{P} \subseteq RE(\{0, 1\})$. Then

$$L_\mathcal{P} = \{w \in \{0, 1\}^* \mid L(M(w)) \in \mathcal{P}\}.$$ 

Theorem 9.11: (Rice’s Theorem) If $\mathcal{P}$ is a nontrivial property of $RE(\{0, 1\})$, then $L_\mathcal{P}$ is not recursive.
Case 1: $\emptyset \notin \mathcal{P}$.

- We will show that $L_U \leq L_{\mathcal{P}}$.
- Because $\mathcal{P}$ is a nontrivial property of $\text{RE}\{\{0,1\}\}$, there exist $L \in \mathcal{P}$, and a TM $M_L$ such that $L(M_L) = L$.
- We will construct a TM $M$ that, on input $x$, produces an output $y$ such that $x \in L_u$ iff $L(M(y)) \in \mathcal{P}$.

$M(y)$ will be a 2-track TM operating as follows on input $w$:

1. Write $x$ to track 2, leaving $w$ on track 1.
2. Return to the left end of $x$ and $w$.
3. Simulate $U$ on $x$ using track 2.
4. If $U$ accepts $x$, simulate $M_L$ on $w$ using track 1.
5. If $M_L$ accepts $w$, accept.

If $x \in L_U$, then $w \in L(M(y))$ iff $w \in L$; i.e., $L(M(y)) = L \in \mathcal{P}$. If $x \notin L_U$, then $L(M(y)) = \emptyset \notin \mathcal{P}$.
How $M$ computes $y$ given $x$:

- Only step 1 of $M(y)$ depends on $x$; hence the remainder of $y$ can be stored in the finite control of $M$.

- $M$ only needs to insert the states and transitions needed to write $x$ on track 2.

Case 2: $\emptyset \in \mathcal{P}$. By Case 1, $L_{\overline{\mathcal{P}}}$ is not recursive.

$$L_{\overline{\mathcal{P}}} = \{w \in \{0, 1\}^* \mid L(M(w)) \in \overline{\mathcal{P}}\}$$

$$= \{w \in \{0, 1\}^* \mid L(M(w)) \notin \mathcal{P}\}$$

$$= \{w \in \{0, 1\}^* \mid L(M(w)) \in \mathcal{P}\}$$

$$= \overline{L_\mathcal{P}}.$$  

Because $\overline{L_\mathcal{P}}$ is not recursive, $L_\mathcal{P}$ is not recursive.