**Defn:** Let $L_{eq}$ be the set of all strings over \( \{0, 1\} \) having an equal number of 0s and 1s.

**Question:** Is $L_{eq}$ regular?

**Theorem 4.1: (The pumping lemma for regular languages)** Let $L$ be a regular language. Then there is an $n \in \mathbb{N}$ such that for every $w \in L$ for which $|w| \geq n$, we can write $w = xyz$ such that:

1. $y \neq \epsilon$;
2. $|xy| \leq n$; and
3. $\forall k \geq 0 : xy^k z \in L$.

**Claim:** $L_{eq}$ is not regular.

**Proof:** By contradiction. Assume $L_{eq}$ is regular. Let $n$ be the constant guaranteed by the pumping lemma. Let $w = 0^n1^n$. Clearly, $w \in L_{eq}$. By the pumping lemma, we can write $w = xyz$ such that $y \neq \epsilon$, $|xy| \leq n$, and $\forall k \in \mathbb{N} : xy^k z \in L_{eq}$. Because $|xy| \leq n$, $y = 0^i$ such that $0 < i \leq n$. Then $xy^0 z = 0^{n-i}1^n \notin L_{eq}$ — a contradiction.
Proof of Pumping Lemma (sketch):

- Let $A = (Q, \Sigma, \delta, q_0, F)$ be a DFA s.t. $L(A) = L$.
- Let $n = |Q|$, $w \in L$, $|w| \geq n$.
- For $0 \leq i \leq n$, let $w_i$ denote the prefix of $w$ with length $i$.
- By the Pigeonhole Principle, 
  $\exists 0 \leq i < j \leq n : \delta(q_0, w_i) = \delta(q_0, w_j)$.
- Let $x = w_i$, $xy = w_j$, $xyz = w$ (note that $|xy| \leq n$, $y \neq \epsilon$).

Proof continued:

- By induction on $k$, $\delta(q_0, xy^k) = \delta(q_0x)$ for all $k \in \mathbb{N}$.

  $\delta(q_0, xy^kz) = \delta(\delta(q_0, xy^k), z)$
  $= \delta(\delta(q_0, x), z)$
  $= \delta(\delta(q_0, xy), z)$
  $= \delta(q_0, xyz)$
  $\in F$.

- Therefore, $xy^kz \in L(A) = L$. 
**Theorem 4.5:** The set of regular languages over $\Sigma$ is closed under complementation; i.e., if $L$ is a regular language over $\Sigma$, then $\Sigma^* - L$ is regular.

**Proof:** Let $A = (Q, \Sigma, \delta, q_0, F)$ be a DFA s.t. $L(A) = L$. Let $A' = (Q, \Sigma, \delta, q_0, Q - F)$. Then

$$\Sigma^* - L = \Sigma^* - \{w \in \Sigma^* \mid \hat{\delta}(q_0, w) \in F\}$$

$$= \{w \in \Sigma^* \mid \hat{\delta}(q_0, w) \in Q - F\}$$

$$= L(A').$$

Thus, $\Sigma^* - L$ is regular.

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**Theorem 4.8:** The set of regular languages over $\Sigma$ is closed under intersection.

**Proof:** Let $L_1$ and $L_2$ be regular languages over $\Sigma$. By DeMorgan’s laws,

$$L_1 \cap L_2 = \overline{L_1 \cup \overline{L_2}}.$$ 

Because the set of regular languages over $\Sigma$ is closed under union and complementation, $L_1 \cap L_2$ is regular.
Defn: Let $\Sigma$ and $\Delta$ be two alphabets. A *homomorphism* from $\Sigma^*$ to $\Delta^*$ is any function $h : \Sigma^* \to \Delta^*$ such that $\forall x, y \in \Sigma^*$, $h(xy) = h(x)h(y)$.

Defn: Let $\Sigma$ and $\Delta$ be two alphabets. A *homomorphism* from $2^{\Sigma^*}$ to $2^{\Delta^*}$ is any function $h : 2^{\Sigma^*} \to 2^{\Delta^*}$ such that

$$h(L) = \{ x \in \Delta^* | \exists y \in L : x = h'(y) \}$$

where $h'$ is a homomorphism from $\Sigma^*$ to $\Delta^*$.

**Theorem 4.14:** Let $h : 2^{\Sigma^*} \to 2^{\Delta^*}$ be a homomorphism. If $L \subseteq \Sigma^*$ is regular, then $h(L)$ is also regular.