Operations on Languages

**Defn:** Let $L_1$ and $L_2$ be two languages. The *concatenation* of $L_1$ and $L_2$ is defined to be

$$L_1 L_2 = \{xy \mid x \in L_1, y \in L_2\}.$$ 

**Defn:** Let $L$ be a language. We define the *Kleene closure of $L$*, denoted $L^*$, to be the least set $S$ such that

- $\epsilon \in S$; and
- if $x \in L$ and $y \in S$, then $xy \in S$.

**Defn:** Let $L \subseteq \Sigma^*$. We recursively define:

- $L^0 = \{\epsilon\}$; and
- $L^{i+1} = LL^i$ for $i \in \mathbb{N}$.

**Defn:** Let $L \subseteq \Sigma^*$. Then

$$L^+ = \bigcup_{i>0} L^i.$$ 

**Claim:** Let $L \subseteq \Sigma^*$. Then

$$L^* = \bigcup_{i \in \mathbb{N}} L^i.$$
Proof sketch:

\[ \subseteq: \] By induction on \( y \in L^*. \)

**Base:** \( y = \epsilon. \) Then \( y \in L^0. \)

**IH:** Assume that for some \( y \in L^*, y \in \bigcup_{i \in \mathbb{N}} L^i. \)

**IS:** Let \( x \in L. \)

- For some \( i, y \in L^i. \)
- Then \( xy \in L^{i+1}. \)

\[ \supseteq: \] We show by induction on \( i \in \mathbb{N} \) that \( L^i \subseteq L^*. \)

**Base:** \( i = 0. \) Then

\[
L^i = \{ \epsilon \}
\subseteq L^*.
\]

**IH:** Assume that for some \( i \in \mathbb{N}, L^i \subseteq L^*. \)

**IS:**

\[
L^{i+1} = LL^i
\]
\[ \subseteq LL^*
\]
\[ \subseteq L^*. \]
Defn: Let $\Sigma$ be an alphabet. We define $R(\Sigma)$ to be the least set $R$ such that

- $\emptyset \in R$;
- $\{\epsilon\} \in R$;
- for each $a \in \Sigma$, $\{a\} \in R$;
- if $L_1 \in R$ and $L_2 \in R$, then $L_1 \cup L_2 \in R$;
- if $L_1 \in R$ and $L_2 \in R$, then $L_1L_2 \in R$; and
- if $L \in R$, then $L^* \in R$.

Regular Expressions

Defn: A regular expression is a description of a language in $R(\Sigma)$ using the following notation:

- $\epsilon$ denotes $\{\epsilon\}$;
- $a$ denotes $\{a\}$ for $a \in \Sigma$; and
- union is denoted by $+$.
Precedence in regular expressions:

1. Kleene closure
2. Concatenation
3. Union

**Theorem 3.4:** Let $L$ be a regular language over $\Sigma$. Then $L \in R(\Sigma)$.

**Lemma 1:** The set of regular languages over $\Sigma$ is closed under union.

**Proof sketch:**
**Lemma 2:** The set of regular languages over $\Sigma$ is closed under concatenation.

**Proof sketch:**

![Diagram for Lemma 2]

**Lemma 3:** The set of regular languages over $\Sigma$ is closed under Kleene closure.

**Proof sketch:**

![Diagram for Lemma 3]
Theorem 3.7: If $L \in R(\Sigma)$, then $L$ is regular.

Proof sketch: By induction on $L \in R(\Sigma)$.

Base: $\emptyset$, $\{\epsilon\}$, and $\{a\}$ for $a \in \Sigma$ are easily seen to be regular.

IH: Assume that some $L_1, L_2 \in R(\Sigma)$ are regular.

IS: By Lemmas 1-3, $L_1 \cup L_2$, $L_1 L_2$, and $L_1^*$ are regular.