**Defn:** A *pushdown automaton* (PDA) is a 7-tuple \((Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)\), where

- \(Q\) is a finite set of *states*;
- \(\Sigma\) is the *input alphabet*;
- \(\Gamma\) is the *stack alphabet*;
- \(\delta : Q \times (\Sigma \cup \{\varepsilon\}) \times \Gamma \to 2^{Q \times \Gamma^*}\) s.t. \(\delta(q, a, X)\) is finite is the *transition function*;
- \(q_0 \in Q\) is the *start state*;
- \(Z_0 \in \Gamma\) is the *start symbol*; and
- \(F \subseteq Q\) is the set of *final states*.

**Defn:** Let \(P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)\) be a PDA. A triple \((q, w, \gamma) \in Q \times \Sigma^* \times \Gamma^*\) is an *instantaneous description* of \(P\). We denote \(Q \times \Sigma^* \times \Gamma^*\) by \(\text{ID}(P)\).

**Defn:** Let \(P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)\) be a PDA. We define the binary relation \(\vdash_P\) on \(\text{ID}(P)\) such that

\[(q, aw, X\beta) \vdash_P (p, w, \alpha\beta) \iff (p, \alpha) \in \delta(q, a, X)\]

for \(p, q \in Q, a \in \Sigma \cup \{\varepsilon\}, w \in \Sigma^*, X \in \Gamma, \alpha, \beta \in \Gamma^*\). \(\vdash_P^*\) is the reflexive transitive closure of \(\vdash_P\).
**Theorem 6.5:** Let \( P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F) \) be a PDA, and suppose \( (q, x, \alpha) \vdash^* (p, y, \beta) \). Then \( (q, xw, \alpha\gamma) \vdash^* (p, yw, \beta\gamma) \) for all \( w \in \Sigma^*, \gamma \in \Gamma^* \).

**Proof:** By induction on \( \vdash^* \).

---

**Defn:** Let \( P = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F) \) be a PDA. The *language accepted by \( P \) by final state* is given by \( L(P) = \{ w \in \Sigma^* : \exists q \in F, \alpha \in \Gamma^* : (q_0, w, Z_0) \vdash^* (q, \epsilon, \alpha) \} \).

The *language accepted by \( P \) by empty stack* is given by \( N(P) = \{ w \in \Sigma^* : \exists q \in Q : (q_0, w, Z_0) \vdash^* (q, \epsilon, \epsilon) \} \).

**Theorem 6.9:** Let \( P_N = (Q, \Sigma, \Gamma, \delta_N, q_0, Z_0, F) \) be a PDA. There is a PDA \( P_F \) such that \( L(P_F) = N(P_N) \).
Proof sketch: Let

\[ P_F = (Q \cup \{p_0, p\}, \Sigma, \Gamma \cup \{X_0\}, \delta_F, p_0, X_0, \{p\}) \]

where \( p_0, p \notin Q, X_0 \notin \Gamma \), and

- \( \delta_F(p_0, \epsilon, X_0) = \{(q_0, Z_0X_0)\} \);
- \( \delta_F(q, a, X) = \delta_N(q, a, X) \forall q \in Q, a \in \Sigma \cup \{\epsilon\}, X \in \Gamma \);
- \( \delta_F(q, \epsilon, X_0) = \{(p, \epsilon)\} \); and
- no other transitions exist in \( P_F \).

\[(p_0, w, X_0) \vdash^*_P (p, \epsilon, \alpha) \]
\[ \iff (q_0, w, Z_0X_0) \vdash^*_P (p, \epsilon, \alpha) \]
\[ \iff (q_0, w, Z_0X_0) \vdash^*_P (q, \epsilon, X_0\alpha) \]
for some \( q \in Q \).

It can be shown by induction on \( \vdash^* \) that for any \( q \in Q, \beta \in \Gamma^*, x, y \in \Sigma^* \),

\[(q_0, xy, Z_0X_0) \vdash^*_P (q, y, \beta X_0\alpha) \]
\[ \iff \alpha = \epsilon \land (q_0, xy, Z_0) \vdash^*_P (q, y, \beta). \]
Therefore, \( w \in L(P_F) \)
\[
\iff \exists \alpha \in (\Gamma \cup \{X_0\})^* : (p_0, w, X_0) \Downarrow_{P_F}^* (p, \epsilon, \alpha)
\]
\[
\iff \exists q \in Q : (q_0, w, Z_0X_0) \Downarrow_{P_F}^* (q, \epsilon, X_0\alpha)
\]
\[
\iff \exists q \in Q : (q_0, w, Z_0) \Downarrow_{P_N}^* (q, \epsilon, \epsilon)
\]
\[
\iff w \in N(P_N).
\]

**Theorem 6.11:** Let
\[
P_F = (Q, \Sigma, \Gamma, \delta_F, q_0, Z_0, F')
\]
be a PDA. Then there is a PDA \( P_N \) such that \( N(P_N) = L(P_F) \).

**Construction:** Let
\[
P_N = (Q \cup \{p_0, p\}, \Sigma, \Gamma \cup \{X_0\}, \delta_N, p_0, X_0, \emptyset)
\]
where \( p_0, p \notin Q \), \( X_0 \notin \Gamma \), and \( \delta_N \) contains all the transitions of \( \delta_F \) plus
\begin{itemize}
  \item \( \delta_N(p_0, \epsilon, X_0) = \{(q_0, Z_0X_0)\} \); and
  \item \( (p, \epsilon) \in \delta_N(q, \epsilon, Y) \) if \( q \in F \cup \{p\} \), \( Y \in \Gamma \cup \{X_0\} \).
\end{itemize}
Derivations

Claim: Let $G = (V, T, P, S)$ be a CFG. Then for $\alpha \in (V \cup T)^*$ and $w \in T^*$, $\alpha \Rightarrow^* w$ iff there exist $x_1, \ldots, x_n \in T^*$, $A_1, \ldots, A_n \in V$, and $\beta_1, \ldots, \beta_n \in (V \cup T)^*$ such that

- $\alpha = x_1 A_1 \beta_1$;
- for $1 \leq i < n$, $x_i A_i \beta_i \Rightarrow x_i \gamma_i \beta_i = x_{i+1} A_{i+1} \beta_{i+1}$; and
- $x_n A_n \beta_n \Rightarrow x_n \gamma_n \beta_n = w$.

Such a sequence is called a leftmost derivation of length $n$.

Theorem 6.13: For any CFL $L$, there is a PDA $M$ such that $N(M) = L$.

Proof sketch: Let $G = (V, T, P, S)$ be a CFG such that $L(G) = L$. Let

$M = (\{q\}, T, V \cup T, \delta, q, S, \emptyset)$, where

- $\delta(q, \epsilon, A) = \{(q, \alpha) \mid A \rightarrow \alpha \in P\} \ \forall A \in V$;
- $\delta(q, a, a) = \{(q, \epsilon)\} \ \forall a \in T$; and
- $\delta(q, a, A) = \emptyset$ otherwise.
For every $\alpha \in (V \cup T)^* \text{, } x \in T^*$,

$$\alpha \Rightarrow x \iff (q, x, \alpha) \rightarrow^* (q, \epsilon, \epsilon).$$

Therefore,

$$N(M) = \{ w \in T^* \mid (q, w, S) \rightarrow^* (q, \epsilon, \epsilon) \}$$

$$= \{ w \in T^* \mid S \Rightarrow^* w \}$$

$$= L(G)$$

$$= L.$$

**Theorem 6.14:** Let $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA. Then $N(M)$ is a CFL.