Stability Preserving Simulations and Bisimulations for Hybrid Systems

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***Abstract*—Pre-orders and equivalence relations between processes, like simulation and bisimulation, have played a central role in the minimization and abstraction based verification and analysis of discrete-state systems for modal and temporal properties. In this paper, we investigate the pre-orders and equivalence relations on hybrid systems which preserve stability. We first show that stability with respect to reference trajectories is not preserved by either the traditional notion of bisimulation or the more recently proposed stronger notions with additional continuity constraints. We introduce the concept of uniformly continuous simulation and bisimulation — namely, simulation and bisimulation with some additional uniform continuity conditions on the relation — that can be used to reason about stability of trajectories. Finally, we show that uniformly continuous simulations and bisimulations are widely prevalent, by recasting many classical results on proving stability of dynamical and hybrid systems as establishing the existence of a simple, obviously stable system that (bi)-simulates the given system through uniformly continuous (bi)-simulations. We also discuss briefly a new abstraction method for stability analysis which is based on the foundations developed in the paper.**

***Index Terms*—Hybrid control systems, formal verification, abstractions, simulations, bisimulations**

I. INTRODUCTION

In this paper, we develop the foundations for simplification based analysis of stability of hybrid systems. Hybrid systems [25] are those systems whose states evolve continuously with real-time, modeling physical processes, while making occasional discrete mode changes, to reflect steps taken by a discrete, digital controller, or operating environment. Such models arise particularly naturally when describing embedded and cyber-physical systems. In such systems, stability is a fundamental requirement. We focus on the notion of Lyapunov and asymptotic stability with respect to reference trajectories as opposed to equilibrium points. The generalization from the equilibrium points to reference trajectories is useful in the setting of hybrid systems. For example, one would expect the behavior of a robot to depend gracefully on its initial orientation.

The basis for scalable automated analysis lies in being able to simplify the system under study for the purpose of analysis. The foundations of such a simplification based analysis is provided by the notions of simulation and bisimulation. Bisimulation [33] is a concept in process algebra that is used to understand when two transition systems are intended to be equivalent. It is taken to be the finest behavioral congruence that one would like to impose. Several correctness specifications of interest in discrete-systems analysis are invariant under bisimulation, i.e., if two systems are bisimilar, then either both satisfy the specification or neither one does. These include safety, reachability and properties expressed in modal and temporal logics such as Hennessy Milner Logic [33], Linear-time Temporal Logic, Computation Tree Logic and μ-calculus [10]. Hence, whether a system satisfies a property is equivalent to that of a bisimulation equivalent system. In fact, finite state bisimulation quotients, can be efficiently computed for finite state systems (with large state-spaces) [31] as well as for subclasses of hybrid systems [3], [30], [7].

While bisimulations define a notion of equivalence between systems, simulations define a pre-order on systems - a reflexive, transitive relation. Simulations preserve properties in one direction given by the pre-order, i.e, if a system simulates another system, then the satisfaction of the property by the former implies the satisfaction of the property by the latter, however, the converse is not necessarily true. Safe fragments of the modal and temporal logics are preserved by the pre-order. Hence, simulations are the basis for abstraction based analysis of infinite state systems [4], [9], [1], wherein an abstract system - a simpler system that ignores some of the details of the given system that may be irrelevant to the satisfaction of the specification - is constructed, and analysed. Approximate notions of simulations and bisimulations have also been introduced and used for simplifying the continuous dynamics and reducing the state-space of standard dynamical and hybrid systems [17], [18], [19], [39] for safety verification. Minimization and simplification techniques are essential in the analysis of systems for reducing the complexity and achieving scalability.

In this paper, we focus on developing notions analogous to simulations and bisimulations for reasoning about stability of hybrid systems. It was observed by Cuijpers in [11] that stability is not bisimulation invariant. Cuijpers strengthened the notion of bisimulation with continuity constraints to show that Lyapunov stability with respect to an equilibrium point is preserved by this stronger notion. Cuijpers' result, unfortunately, does not extend when one considers asymptotic stability with respect to an equilibrium point or Lyapunov or asymptotic stability with respect to a reference trajectory. We illustrate this through counter-examples in Section VII. We, therefore, identify congruences and pre-orders that allow one to reason about stability of trajectories. Our main observation is that in this case, *uniform* continuity conditions must be imposed on simulation and bisimulation relations. Thus, we introduce the notions of *uniformly continuous bisimulation* and *uniformly continuous simulation*. We show that stability

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(Lyapunov or asymptotic) of trajectories is invariant under the new notion of bisimulation. Moreover, we show that uniformly continuous simulations yield the right notion of abstraction for stability — if D_1 is uniformly simulated by D_2 and D_2 is stable then D_1 is also stable.

Next, we examine if the preorders developed in the paper can serve as the basis for abstraction based stability analysis. We provide preliminary results towards the applicability of the definitions by casting several classical techniques in control theory and hybrid systems for stability analysis as simplification based analysis techniques in which the simplified system is related to the original system by uniformly continuous simulation or bisimulation. First, we examine the Lyapunov's first method for stability analysis of continuous dynamical systems based on linearization. The Hartman-Grobman theorem [22], [23], [24] establishes that the behavior of any dynamical system near a hyperbolic equilibrium point is topologically the same as the behavior of the linear system near the same equilibrium point. We observe that, in fact, the Hartman-Grobman theorem establishes that there is a uniformly continuous bisimulation between the dynamical system and its linearization (in a small neighborhood around the equilibrium point). Next, we consider Lyapunov's second method and its extensions based on multiple Lyapunov functions for switched and hybrid systems [29], [32]. A Lyapunov function (in a small neighborhood around the equilibrium) is essentially a uniformly continuous simulation between the given system and a one dimensional system (image of the given system with respect to the Lyapunov function), which is trivially stable (its value continuously decreases). Similarly, one can interpret the multiple Lyapunov functions based technique as constructing a simpler stable abstraction via the multiple Lyapunov functions that uniformly continuously simulates the original system. Finally, we discuss some recent results on finite abstraction based techniques for stability analysis [38], [36] motivated by the investigations of the current paper.

A preliminary version of the paper appeared in [35] as a peer-reviewed conference paper. This version contains detailed motivation for the work in the paper (see Section II and Section VI), relevant explanations to improve the readability and fleshes out all the missing proofs. Results showing that uniformly continuous simulations form a pre-order, were not present in [35].

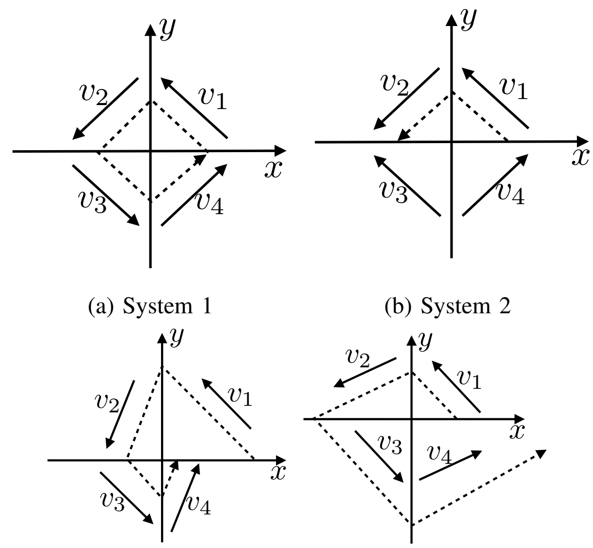
II. Motivation

The line of work in this paper is motivated by the need for formal foundations for an "abstraction based analysis" approach for stability analysis. In this section, we explain this approach and its merit in the context of safety verification, and highlight the significance of the results in the paper.

Abstraction based analysis refers to a two step analysis paradigm. First, a simplification of a given (orignal or concrete) system called the "abstract system" is constructed. Then analysis is performed on the abstract system to deduce the correctness of the original system. We illustrate this approach for safety verification.

Consider the System 1 shown in Figure 1a. Here, the state-space is partitioned into four regions (corresponding to four quadrants) and each of the regions is associated with a differential equation which is given by the vector in the region. For instance, the vector $\mathbf{v} = (-1, 1)$ in the first quadrant (positive x, positive y) corresponds to the differential equation $\dot{\mathbf{z}} = \mathbf{v}\|\mathbf{z}\|$, where $z = (x, y)$ refers to the two dimensional state, and $\|\mathbf{z}\|$ refers to some norm of $\mathbf{z}$. An execution of the system starting at a point follows a trajectory corresponding to a solution of the differential equation of region it is in, until it reaches the boundary, upon which it follows a solution of the differential equation associated with the new region, and so on. An example execution of System 1 is shown by dotted lines in Figure 1a. Essentially, the executions move in the direction given by the arrows in a region, and the "rate" at which they evolve depends on their distance to the origin (closer to the origin implies slower evolution).

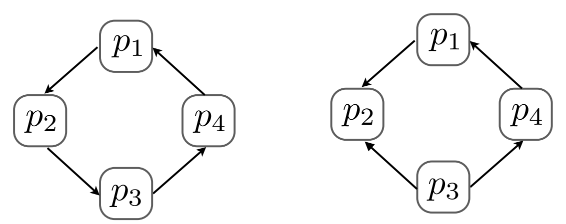
A safety specification provides a set of bad states, and safety verification consists of checking if all the executions starting from an initial state never reach a bad state. In System 1, let us consider the set of points F on the negative y-axis to be the bad states, and the set of points I on the positive x-axis to be the initial states. We intend to verify if F is reachable from an execution starting in I.



(a) System 1 (b) System 2

(c) System 3 (d) System 4

Fig. 1: Examples of Hybrid Systems



(a) $\mathcal{G}_1$: abstract graph of Systems 1, 3 and 4 (b) $\mathcal{G}_2$: abstract graph of System 2

Fig. 2: Examples of abstract graphs

A well-known technique for safety verification is *predicate abstraction*, which constructs a finite state system by using a set of predicates which partition the state-space into a finite number of regions. The regions represent the nodes of the abstract graph, and an edge between two regions indicates the possible existence of an execution which starts at the source region and ends at the target region. In Figure 2a, we present a finite state abstraction $\mathcal{G}_1$ of System 1 from Figure 1a. Here, the nodes p_1, p_2, p_3 and p_4 correspond to the positive y-axis, negative x-axis, negative y-axis and positive x-axis, respectively. An edge from p_i to p_j represents that there is possibly an execution from the region represented by p_i to the region represented by p_j evolving through the quadrant containing p_i and p_j. This is a slight modification of a standard predicate abstraction procedure [21] to illustrate the points in this paper.

The abstract graph "over-approximates" the behaviors of the concrete system. In particular, if there is an execution in the concrete system which passes through a sequence of regions $\pi = p_{i_1} p_{i_1} \ldots$, then π is also a path in the graph. This property is guaranteed through the notion of a simulation which specifies that every trajectory or transition in the original system can be matched by a corresponding element of the abstract system (see Section VI for precise definitions). Hence, if the abstract system does not have a path from $I = p_4$ to $F = p_3$, then the original system is safe. On the other hand, in general, the abstract graph could contain more executions. This is because, the existence of an execution from region 1 to region 2 and an execution from region 2 to regions 3 does not imply that there is an execution corresponding to the path $1, 2, 3$. However, in this particular example, System 1 and abstract graph 1 are "equivalent" or "bisimilar"; hence, we infer that System 1 is unsafe from the fact that $\mathcal{G}_1$ has a path form I to F. Consider, instead, System 2 shown in Figure 1b and its abstract graph $\mathcal{G}_2$ shown in Figure 2b. Note that, in the latter, there is no path from I to F, hence, we can conclude that System 2 is safe, from the fact that the abstract graph $\mathcal{G}_2$ simulates System 2.

Abstractions are crucial for scalable automated verification. Firstly, the concrete system may be too complicated to analyse directly (for instance, falls into an undecidable class for safety verification). Secondly, the state-space of the concrete system may be too large and hence, direct verification practically infeasible. On the other hand, abstractions are often simpler systems for which automated verification is computationally less expensive. However, abstract systems are often conservative, and hence, abstraction based techniques are generally coupled with techniques to iteratively improve the abstractions, for instance, using counter-examples [2], [8].

Simulation relations are the foundations for abstraction based analysis of safety. They specify a relation that needs to hold between a concrete system and its simplification for safety preservation. Also, they define a pre-order on the space of abstract systems, and hence, provide the basis for refinements. The objective of this paper is to explore the foundations for abstraction based analysis of stability. Note that the abstraction technique described in this section is not very useful for reasoning about stability. For instance, System 1 and System 3 are stable (Lyapunov and asymptotically stable, respectively), and System 4 is unstable with respect to the origin. However, the abstract graph corresponding to all the three systems is given by $\mathcal{G}_1$ in Figure 2a. This suggests that stability analysis requires new abstraction techniques. In this paper, we address a fundamental question and a guiding principle towards the development of such techniques "what relation between a concrete system and an abstract system preserves stability"?

III. Preliminaries

Let $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ denote the set of reals and non-negative reals, respectively. Let $\mathbb{R}_\infty$ denote the set $\mathbb{R}_{\geq 0} \cup \{\infty\}$, where ∞ denotes the largest element of $\mathbb{R}_\infty$, that is, $x < \infty$ for all $x \in \mathbb{R}_{\geq 0}$. Also, for all $x \in \mathbb{R}_\infty$, $x + \infty = \infty$. Let $\mathbb{N}$ denote the set of all natural numbers $\{0, 1, 2, \ldots\}$, and let $[n]$ denote the first n natural numbers, that is, $[n] = \{0, 1, 2, \ldots, n-1\}$. Let *Int* denote the set of all closed intervals of the form $[0, T]$, where $T \in \mathbb{R}_{\geq 0}$, and the infinite interval $[0, \infty)$. The *size* of an interval $I \in$ *Int* is T if $I = [0, T]$ and ∞ if $I = [0, \infty)$. We denote the size of an interval $I \in$ *Int* by *Size*(I).

a) Sequences: A sequence σ is a function whose domain is either $[n]$, for some $n \in \mathbb{N}$, or the set of natural numbers $\mathbb{N}$. Length of a sequence σ, denoted $|\sigma|$, is n, if $Dom(\sigma) = [n]$, and is ∞, otherwise. Given a sequence $\sigma : \mathbb{N} \to \mathbb{R}$ and an element r of $\mathbb{R}_\infty$, we use $\sum_{i=0}^{\infty} \sigma(i) = r$ to denote the standard limit condition $\lim_{N \to \infty} \sum_{i=0}^{N} \sigma(i)$.

b) Extended Metric Space: An *extended metric space* is a pair (M, d), where M is a set and $d : M \times M \to \mathbb{R}_\infty$ is a distance function, such that for all m_1, m_2 and m_3,

1) (Identity of indiscernibles) $d(m_1, m_2) = 0$ if and only if $m_1 = m_2$.
2) (Symmetry) $d(m_1, m_2) = d(m_2, m_1)$.
3) (Triangle inequality) $d(m_1, m_3) \leq d(m_1, m_2) + d(m_2, m_3)$.

In the sequel, by a metric space, we mean an extended metric space. Further, when the metric on M is clear, we will simply refer to M as a metric space.

Let us fix an extended metric space (M, d) for the rest of this section. We define an open ball of radius ϵ around a point x to be the set of all points which are within a distance ϵ from x. Formally, an *open ball* is a set of the form $B_\epsilon(x) = \{y \in M \,|\, d(x, y) < \epsilon\}$. An *open set* is a subset of M which can be expressed as a union of open balls. Given a set $X \subseteq M$, a *neighborhood* of X is an open set in M which contains X. Given a subset X of M, an ϵ-neighborhood of X is the set $B_\epsilon(X) = \bigcup_{x \in X} B_\epsilon(x)$. A subset X of M is *compact* if for every collection of open sets $\{U_\alpha\}_{\alpha \in A}$ such that $X \subseteq \bigcup_{\alpha \in A} U_\alpha$, there is a finite subset J of A such that $X \subseteq \bigcup_{i \in J} U_i$.

IV. Set-Valued Functions

We will need set-valued functions and a notion of continuity on these functions. We choose not to treat set-valued functions as single valued functions whose co-domain is a power set, since, as argued in [27], it leads to strong notions of continuity, which are not satisfied by many functions. A *set-valued function* $F : A \rightsquigarrow B$ is a function which maps every element of A to a set of elements in B. We use $Dom(F)$ to denote the domain A of F. Given a set $A' \subseteq A$, $F(A')$ will denote the set $\bigcup_{a \in A'} F(a)$. Given a binary relation $R \subseteq A \times B$, we use R also to denote the set-valued function $R : A \rightsquigarrow B$ given by $R(x) = \{y \,|\, (x, y) \in R\}$. Further, the inverse of a set-valued function $F : A \rightsquigarrow B$, namely, $F^{-1} : B \rightsquigarrow A$, will denote the set-valued function which maps $b \in B$ to the set $\{a \in A \,|\, b \in F(a)\}$. Also, the composition of two set-valued functions $F : A \rightsquigarrow B$ and $G : B \rightsquigarrow C$, is the function $(G \circ F) : A \rightsquigarrow C$, and is defined as $(G \circ F)(a) = \{c \,|\, \exists b \in B, b \in F(a), c \in G(b)\}$. When $F : A \rightsquigarrow B$ is a single-valued function, that is $F(a)$ is a singleton for every $a \in Dom(F)$, we use $F : A \to B$ to denote the function.

c) Continuity of Set-Valued Functions: Let $F : A \rightsquigarrow B$ be a set-valued function, where A and B are metric spaces. First, we define the notion of upper semi-continuity of F, which is a generalization of the "ϵ, δ - definition" of continuity for single valued functions [27].

Definition 1. A function $F : A \rightsquigarrow B$ is said to be *upper semi-continuous* if

$$\forall \epsilon > 0, \forall a \in Dom(F), \exists \delta > 0 \, s.t. \, F(B_\delta(a)) \subseteq B_\epsilon(F(a)).$$

Note that we do not explicitly state the distance functions on the metric spaces used. The open balls are with respect to the distance metrics on the underlying sets. In general, when there is no ambiguity, we will use d for the distance function. Below, we define the notion of lower semi-continuity.

Definition 2. A function $F : A \rightsquigarrow B$ is said to be *lower semi-continuous* if for every open set $V \subseteq B$, the set $\{a \in A \,|\, F(a) \cap V \neq \emptyset\}$ is open.

A function which is both upper and lower semi-continuous is said to be *continuous*. Next, we define a "uniform" version of upper semi-continuity, where, analogous to the case of single valued functions, corresponding to an ϵ, there exists a δ which works for every point in the domain.

Definition 3. A function $F : A \rightsquigarrow B$ is said to be *uniformly upper semi-continuous* if and only if

$$\forall \epsilon > 0, \exists \delta > 0 \, s.t. \, \forall a \in Dom(A), F(B_\delta(a)) \subseteq B_\epsilon(F(a)).$$

We will use the predicate $UC(F, \delta, \epsilon)$ to state that $\forall a \in Dom(A), F(B_\delta(a)) \subseteq B_\epsilon(F(a))$.

Proposition 1. *If $F : A \rightsquigarrow B$ is a uniformly upper semi-continuous function, then F is also a lower semi-continuous function.*

Since the two notions of upper and lower semi-continuity coincide with the notion of continuity, with the addition of uniformity condition on upper semi-continuity, we refer to uniform upper semi-continuity as just uniform continuity.

Proposition 2. *Let $F : A \rightsquigarrow B$ and $G : B \rightsquigarrow C$ be two set-valued uniformly continuous functions. Then: $G \circ F$ is a uniformly continuous function.*

V. HYBRID SYSTEMS

Hybrid systems are systems exhibiting mixed discrete-continuous behaviors. Such behaviors are prevalent in embedded control systems where a digital processor is employed to control, for example, a continuous plant. A popular formalism for modeling hybrid systems is that of *hybrid automata* [25], in which the continuous dynamics is modelled by a system of differential or difference equations, and the discrete control is modelled by a finite state automaton. In this exposition, we will not concern ourselves with any particular representation of these systems, but will use a generic semantic model with trajectories modeling continuous evolution and transitions modeling discrete jumps in the system.

A. Hybrid Transition Systems

Hybrid transition system is an extension of transition system with trajectories. We first define the two components of a hybrid transition system, namely, trajectories and transitions.

Definition 4. Given a set S, a *trajectory* over S is a function $\tau : I \to S$, where $I \in$ *Int* is an interval starting at 0. Let *Traj*(S) denote the set of all trajectories over S.

Definition 5. A *transition* over a set S is a pair $\alpha = (s_1, s_2) \in S \times S$. Let *Trans*$(S)$ denote the set of all transitions over S.

Definition 6. A *hybrid transition system* (*HTS*) $\mathcal{H}$ is a tuple (S, Σ, Δ), where S is a set of states, $\Sigma \subseteq$ *Trans*(S) is a set of transitions and $\Delta \subseteq$ *Traj*(S) is a set of trajectories.

A hybrid transition system where the set of trajectories Δ is empty is referred to as a transition system. Often, the semantics of a hybrid system is expressed in terms of a transition system (either with times annotating the transitions or time-abstract as in here). Such semantics capture enough information about the hybrid system to reason about properties such as reachability. However, for stability, the evolution of the state with respect to time is essential and adding trajectories is a clean way to capture the same.

Remark 1. Trajectories resulting from a differential equation are continuous and differentiable. The hybrid transition system could potentially have deadlock state, i.e., a state starting from which there are no trajectories or transitions. However, we do not impose these restrictions, since they are not important for the correctness of the results, in that, our results will hold even if we require the hybrid transitions system to contain only trajectories that are continuous and differentiable, and not contain deadlock states.

B. Executions and related concepts

Next, we define an execution of a hybrid transition system. We will need the notions of first and last elements of transitions and trajectories. For a trajectory τ, *First*$(\tau) = \tau(0)$, and *Last*(τ) is defined only if $Dom(\tau)$ is a finite interval in which case *Last*$(\tau) = \tau(T)$, where $T =$ *Size*$(Dom(\tau))$. For a transition $\alpha = (s_1, s_2)$, *First*$(\alpha) = s_1$ and *Last*$(\alpha) = s_2$. An execution is a finite or infinite sequence of trajectories and transitions which have matching end-points.

Definition 7. An *execution* of $\mathcal{H}$ is a sequence $\sigma : D \to \Sigma \cup \Delta$, where $D = [n]$ for some $n \in \mathbb{N}$ or $D = \mathbb{N}$, such that for each $0 \leq i < |\sigma| - 1$, $\mathit{Last}(\sigma(i)) = \mathit{First}(\sigma(i+1))$. Let $\mathit{Exec}(\mathcal{H})$ denote the set of all executions of $\mathcal{H}$.

In particular, this implies that all trajectories in an execution, except possibly the last, have finite domain. Given an execution σ, the first state of σ, denoted $\mathit{First}(\sigma)$, is $\mathit{First}(\sigma(0))$. We will denote the set of states appearing in an execution σ as $\mathit{States}(\sigma)$. For a transition α, $\mathit{States}(\alpha) = \{\mathit{First}(\alpha), \mathit{Last}(\alpha)\}$, for a trajectory $\tau \in \Delta$, $\mathit{States}(\tau) = \{\tau(t) \,|\, t \in \mathit{Dom}(\tau)\}$, and for an execution σ, $\mathit{States}(\sigma) = \bigcup_{i \in \mathit{Dom}(\sigma)} \mathit{States}(\sigma(i))$. We extend the functions $\mathit{First}, \mathit{Last}$ and States to a set of trajectories, transitions and executions in the natural way.

In order to define distance between executions, we interpret an execution as a set which we call the graph of the execution. A graph of an execution consists of triples (t, i, x) such that x is a state that is reached after time t has elapsed along the execution, and i is the number of discrete transitions that have taken place before time t. Let us first extend the definition of *Size* to trajectories and transitions. For $\tau \in \mathit{Traj}(S)$, $\mathit{Size}(\tau) = \mathit{Size}(\mathit{Dom}(\tau))$, and for $\alpha \in \mathit{Trans}(S)$, $\mathit{Size}(\alpha) = 0$.

Definition 8. For an execution σ and $j \in \mathit{Dom}(\sigma)$, let $T_j = \sum_{k=0}^{j-1} \mathit{Size}(\sigma(k))$ and $K_j = |\{k \,|\, k < j,\ \sigma(k) \text{ is a transition}\}|$. The *graph* of an execution σ, denoted, $\mathit{Gph}(\sigma)$, is the set of all triples (i, t, x) such that there exists $j \in \mathit{Dom}(\sigma)$ satisfying the following:

- $t \in [T_j, T_j + \mathit{Size}(\sigma(j))]]$.
- If $\sigma(j)$ is a trajectory, then $i = K_j$ and $x = \sigma(j)(t - T_j)$.
- If $\sigma(j)$ is a transition, then either $i = K_j$ and $x = \mathit{First}(\sigma)$, or $i = K_j + 1$ and $x = \mathit{Last}(\sigma)$.

C. Relations on Hybrid Transitions Systems and their elements

Let $\mathcal{H} = (S, \Sigma, \Delta)$ be a hybrid transition system and $g : S \rightsquigarrow S'$ be a set-valued function whose domain is the state-space of $\mathcal{H}$. We extend g to be a set-valued function from $\mathit{Traj}(S)$ to $\mathit{Traj}(S')$ and from $\mathit{Trans}(S)$ to $\mathit{Trans}(S')$ as follows. Given a trajectory $\tau \in \mathit{Traj}(S)$, $g(\tau)$ is the set of trajectories τ' such that domain $\mathit{Dom}(\tau') = \mathit{Dom}(\tau)$ and $\tau'(t) \in g(\tau(t))$ for all $t \in \mathit{Dom}(\tau)$. Similarly, for a transition $\alpha = (s_1, s_2) \in \mathit{Trans}(S)$, $g(\alpha) = \{(s_1', s_2') \,|\, s_1' \in g(s_1), s_2' \in g(s_2)\}$. Also, for an execution σ of $\mathcal{H}$, $g(\sigma)$ is the set of all σ' such that $\mathit{Dom}(\sigma') = \mathit{Dom}(\sigma)$ and for each $i \in \mathit{Dom}(\sigma)$, $\sigma'(i) \in g(\sigma(i))$. If g is a single valued function, then we use $g(\tau)$, $g(\alpha)$ and $g(\sigma)$ to denote the unique element mapped by g. We define $g(\mathcal{H})$ to be the HTS obtained by applying g component-wise, that is, $g(\mathcal{H}) = (g(S), g(\Sigma), g(\Delta))$.

D. Metric Hybrid Transition Systems

A metric hybrid transition system is a hybrid transition system whose set of states is equipped with a metric. A *metric hybrid transition system* (*MHS*) is a pair $(\mathcal{H}, d)$, where $\mathcal{H} = (S, \Sigma, \Delta)$ is a hybrid transition system, and (S, d) is an extended metric space. From now on, we assume that by a hybrid transition system, we mean a metric hybrid transition system. Again, we will abuse notation and always use d for the distance function.

We lift the metric on the state-space to the executions of a system, which will be used later to define stability. Recall that given a metric space (M, d), the *Hausdorff distance* between two sets $A, B \subseteq M$, also denoted $d(A, B)$, is given by

$$\max\{\sup_{p \in A} \inf_{q \in B} d(p, q), \sup_{p \in B} \inf_{q \in A} d(p, q)\}.$$

Definition 9. Let $(\mathcal{H}, d)$ be a metric transition system with $\mathcal{H} = (S, \Sigma, \Delta)$. For $(t_1, i_1, x_1), (t_2, i_2, x_2) \in \mathbb{R}_{\geq 0} \times \mathbb{N} \times S$, let

$$d((t_1, i_1, x_1), (t_2, i_2, x_2)) = \max\{|t_1 - t_2|, |i_1 - i_2|, d(x_1, x_2)\}.$$

The *distance between executions* $\sigma_1, \sigma_2 \in \mathit{Exec}(\mathcal{H})$, denoted as $d(\sigma_1, \sigma_2)$, is defined as,

$$d(\mathit{Gph}(\sigma_1), \mathit{Gph}(\sigma_2)).$$

The above definition of distance between two hybrid executions is borrowed from [20]. Note that in the above definition, for $\epsilon < 1$, $i_1 - i_2$ is always 0. Hence, for two executions to be "close", the above definition requires that the states in i-the discrete transition in one execution are close to those of the i-th transition in the other. However, it does not require that the states in the two executions at exactly the same time be close, but only requires them to be close at nearby times. Hence, it allows for a wiggle in the times at which discrete transitions are taken in the two executions.

Two executions are said to converge, if the distance between the two decreases as we consider suffixes of the executions starting from larger and larger times. Given a subset G of $\mathbb{R}_{\geq 0} \times \mathbb{N} \times S$ and a $T \in \mathbb{R}_{\geq 0}$, let us denote by $G|_T$ the set $\{(t, i, x) \in G \,|\, t \geq T\}$.

Definition 10. Two executions σ_1 and σ_2 are said to *converge* if for every real $\epsilon > 0$, there exists a time $T \in \mathbb{R}_{\geq 0}$ such that $d(\mathit{Gph}(\sigma_1)|_T, \mathit{Gph}(\sigma_2)|_T) < \epsilon$.

We will use the predicate $\mathit{Conv}(\sigma_1, \sigma_2)$ to denote the fact that σ_1 and σ_2 converge.

Remark 2. We choose to use the definition of graphical distance to define distance between executions. There are alternate definitions of distance on executions of hybrid systems which have been investigated in the literature, including the Skorokhod metrics and the generalized Skorokhod metrics (see, for example, [14]). These definitions define the distance between executions using a retiming map between the time domains of executions, such that, the distance is the infimum over all retiming maps of the maximum distance between the values of times mapped by the retiming map and between the states at those times.

Also, there might be situations, where, from the point of view of stability, one might want to ignore the effects of the discrete transitions, as these changes happen on a set of "measure 0" in the time domain. More precisely, the distance between two executions can be defined as the pointwise distance between the functions obtained from the executions by first dropping all the (discrete) transitions and then concatenating all the trajectories in order.

Though we choose to use graphical distance as the notion of distance between executions in our technical development, our results hold for all the definitions of the distance discussed above.

E. Stability of Hybrid Transition Systems

Now, we introduce the formal definitions of Lyapunov and asymptotic stability with respect to reference executions. Informally, stability capture the notion that small perturbations in the initial state of the system results in only small variations in the resulting behaviors.

d) Lyapunov Stability: We first define the notion of Lyapunov stability. Given an HTS $\mathcal{H}$ and a set of executions $\mathcal{T} \subseteq \mathit{Exec}(\mathcal{H})$, we say that $\mathcal{H}$ is *Lyapunov stable* (*LS*) with respect to $\mathcal{T}$, if for every $\epsilon > 0$ in $\mathbb{R}_{\geq 0}$, there exists a $\delta > 0$ in $\mathbb{R}_{\geq 0}$ such that the following condition holds:

$$\forall \sigma \in \mathit{Exec}(\mathcal{H}), \mathit{First}(\sigma(0)) \in B_\delta(\mathit{First}(\mathcal{T})) \implies \exists \rho \in \mathcal{T}, d(\sigma, \rho) < \epsilon. \tag{1}$$

The above statement says that for every execution σ of the system $\mathcal{H}$ which starts within a distance δ of the initial state of some execution in $\mathcal{T}$, there exists an execution ρ in $\mathcal{T}$ which is within distance ϵ from σ. We will use a predicate $\mathit{Lyap}(\mathcal{H}, \mathcal{T}, \delta, \epsilon)$ which evaluates to true when Equation 1 holds.

e) Asymptotic Stability: Next, we define asymptotic stability which, in addition to Lyapunov stability, requires convergence of trajectories in some neighborhood around the reference execution. An HTS $\mathcal{H}$ is said to be *asymptotically stable* (*AS*) with respect to a set of execution $\mathcal{T} \subseteq \mathit{Exec}(\mathcal{H})$, if it is Lyapunov stable and there exists a $\delta > 0$ in $\mathbb{R}_{\geq 0}$ such that

$$\forall \sigma \in \mathit{Exec}(\mathcal{H}), \mathit{First}(\sigma(0)) \in B_\delta(\mathit{First}(\mathcal{T})) \implies \exists \rho \in \mathcal{T}, \mathit{Conv}(\sigma, \rho). \tag{2}$$

So, a system $\mathcal{H}$ is asymptotically stable with respect to a subset of its executions $\mathcal{T}$, if $\mathcal{H}$ is Lyapunov stable with respect to $\mathcal{T}$, and every execution starting within a distance of δ from the starting point of some execution in $\mathcal{T}$ converges to some execution in $\mathcal{T}$.

Remark 3. The notions of stability with respect to an equilibrium point are a special case of the notion of stability with respect to trajectories, since an equilibrium point is essentially a special trajectory which always remains at the equilibrium point.

VI. Simulations and Bisimulations

In this section, we define the notions of simulation and bisimulation and provide a brief overview of the notions in the context of formal verification.

Our definitions of simulation and bisimulation between hybrid transition systems is along the lines of [28]. Intuitively, a simulation between two systems defines a binary relation on their state-space, such that every trajectory or transition in one system is matched by a trajectory or transition of the other system. This local condition however ensures that every execution in one system is related to some execution in the other system by the binary relation on the state-space.

Definition 11. Given two hybrid transition systems $\mathcal{H}_1 = (S_1, \Sigma_1, \Delta_1)$ and $\mathcal{H}_2 = (S_2, \Sigma_2, \Delta_2)$, a binary relation $R \subseteq S_1 \times S_2$ is said to be a *simulation relation* from $\mathcal{H}_1$ to $\mathcal{H}_2$, denoted $\mathcal{H}_1 \preceq_R \mathcal{H}_2$, if for every $(s_1, s_2) \in R$, the following conditions hold:

- for every state s_1' such that $(s_1, s_1') \in \Sigma_1$, there exists a state s_2' such that $(s_2, s_2') \in \Sigma_2$ and $(s_1', s_2') \in R$; and
- for every trajectory $\tau_1 \in \Delta_1$ such that $\mathit{First}(\tau_1) = s_1$, there exists a trajectory $\tau_2 \in \Delta_2$ such that $\mathit{First}(\tau_2) = s_2$, and $\tau_2 \in R(\tau_1)$.

Note that if $(s_1, s_2) \in R$, then for every execution σ starting from s_1, there exists an execution σ_2 starting from s_2 such that σ_1 and σ_2 are related by R. Hence, $\mathcal{H}_2$ has "more behaviors" than $\mathcal{H}_1$. We say that $\mathcal{H}_1$ is similar to $\mathcal{H}_2$, denoted $\mathcal{H}_1 \preceq \mathcal{H}_2$, if there is a simulation relation R from $\mathcal{H}_1$ to $\mathcal{H}_2$. Simulation is a pre-order on the set of hybrid transition systems, that is, $\preceq$ is reflexive ($\mathcal{H} \preceq \mathcal{H}$) and transitive ($\mathcal{H}_1 \preceq \mathcal{H}_2$ and $\mathcal{H}_2 \preceq \mathcal{H}_3$ implies $\mathcal{H}_1 \preceq \mathcal{H}_3$).

Definition 12. A binary relation $R \subseteq S_1 \times S_2$ is a *bisimulation relation* between $\mathcal{H}_1$ and $\mathcal{H}_2$, denoted $\mathcal{H}_1 \sim_R \mathcal{H}_2$, if R is a simulation relation from $\mathcal{H}_1$ to $\mathcal{H}_2$ and R^{-1} is a simulation relation from $\mathcal{H}_2$ to $\mathcal{H}_1$.

We will use $\mathcal{H}_1 \sim \mathcal{H}_2$ to denote the fact that $\mathcal{H}_1$ and $\mathcal{H}_2$ are bisimilar, that is, there exists a bisimulation relation R between $\mathcal{H}_1$ and $\mathcal{H}_2$. Note that bisimulation $\sim$ is an equivalence relation on the set of hybrid transition systems.

Bisimulation is the canonical notion of congruence studied in the context of process algebra [33]. It defines when two processes (transition systems) are behaviorally equivalent, namely, exhibit the same set of sequences of observations, are equivalent with respect to reaching a deadlock state, and are equivalent when a subprocess is replaced by an equivalent process. Weaker notions such as equivalence with respect to the set of sequences of observations alone do not satisfy the above conditions. Further, bisimulation is a powerful notion of equivalence between processes, since it is known to preserve a large set of interesting discrete-time properties. This includes properties expressed in modal and temporal logics such as Hennessy Milner Logic, Linear-time Temporal Logic (LTL), Computation Tree Logic (CTL*) and μ-calculus, which express linear-time and branching-time properties of transition systems [33], [10]. These logics contain operators which can quantify over some branch or all branches of the process. For instance, they can express properties such as there exists an execution of the system in which there exists a state such that all execution starting from the state satisfy some other property. Bisimulation preserves the above properties in the sense that if two systems are bisimilar, then either both satisfy the property or both do not satisfy the property. Hence, such properties are referred to as bisimulation invariant properties.

Bisimulation is the main ingredient in minimization and verification of bisimulation invariant properties. Minimization consists of constructing a minimal system which is bisimilar to the given system. It suffices to verify if the minimal system satisfies such a property to infer whether the given system satisfies the property. More generally, constructing a "smaller" or "simpler" system which is bisimilar to the original system can result in a reduction in the verification effort. In fact, for infinite state systems such as hybrid systems, several verification algorithms rely on constructing finite state systems bisimilar to the hybrid systems, for which the property verification can be efficiently performed. For instance, for the subclass of hybrid systems called timed automata [3], which consist of variables (referred to as clocks) evolving at a unit rate, the behavior of the system from a state depends only on the the integer values between which each of the variables lie (up to a maximum integer constant given by the system description) and the ordering between the fractional values of the variables. Hence, the infinite state-space is partitioned into "regions" each of which consist of states which are equivalent with respect to the above constraints on the variable values. A finite state system constructed using the regions as the nodes is bisimilar to the original timed automaton. Similar observations are used to construct finite state systems bisimilar to several subclasses of hybrid systems including initialized rectangular hybrid automata [26], o-minimal systems [30], [7] and STORMED systems [40].

The classes of hybrid systems which are bisimilar to finite state systems is limited. Hence, in general, one relies on a conservative analysis using simulations. Simulations preserve several classes of properties in one direction. For instance, consider the problem of safety verification: given a set of safe states, does every execution of the system only contain safe states. Suppose that $\mathcal{H}_1$ is simulated by $\mathcal{H}_2$ (both have the same state-space and the simulation relation is identity) and $\mathcal{H}_2$ contains only the safe states. Then one can conclude that $\mathcal{H}_1$ also contains only safe states, since every execution in $\mathcal{H}_1$ has a matching execution (the same execution) in $\mathcal{H}_2$ which contains only safe states. More generally, the universal fragments of modal logic, CTL* and μ-calculus are preserved by simulation. Intuitively, if all the executions of $\mathcal{H}_2$ satisfy some property (such as remaining within the safe set), then all the executions of $\mathcal{H}_1$ satisfy the property. However, the same is not true for existential quantification, in that, if there exists an execution of $\mathcal{H}_2$ which satisfies certain properties, then there may not be an execution of $\mathcal{H}_1$ which satisfies the property.

Simulations are the basis for abstraction based methods for system analysis. Here, a simpler abstract system which simulates the original system is constructed and the abstract system is analyzed. For instance, in predicate abstraction [21], discussed in Section II, a finite abstract graph is constructed by dividing the state-space into a finite number of regions each of which correspond to a node in the graph and adding an edge between two nodes if there exists an execution in the original system between the corresponding regions. Such a system, in general, only simulates the original system, since bisimulation would require that when there exists an edge between two nodes, there exists an execution in the original system from *every* state in the region corresponding to the source node to some state in the region corresponding to the target node. Note that the satisfaction of the property by the abstract graph implies the satisfaction of the property by the original system, however, if the abstract graph does not satisfy the property (preserved by simulation), then it does not imply that the original system does not satisfy the property. Hence, to account for this conservativeness, the abstraction based methods are often accompanied by an abstraction refinement loop. These methods iteratively construct less conservative systems according to the simulation pre-order, that is, if $\mathcal{H}_2$ simulates $\mathcal{H}_1$ and $\mathcal{H}_2$ fails to satisfy the property, then the method constructs a system $\mathcal{H}_3$ such that $\mathcal{H}_2$ simulates $\mathcal{H}_3$ and $\mathcal{H}_3$ simulates $\mathcal{H}_1$. There are several methods for constructing such refinements, one popular technique is to construct refinements based on a counter-example [2], [8], a witness to the violation of the property by the previous abstraction. There are also methods which construct abstractions which are not finite state systems, but systems with simpler discrete or continuous dynamics [5], [34].

VII. Uniformly Continuous Relations and Stability Preservation

The main focus of the paper is to investigate the right pre-orders on hybrid systems required to reason about stability properties. Bisimulations have been the canonical notion of congruence for reasoning about a large fraction of interesting properties in the discrete setting. Properties expressed in modal and temporal logics such as LTL, CTL and μ-calculus are invariant under bisimulation; that is, all systems which are bisimilar satisfy the same set of properties.

Stability requires reasoning about the distance between trajectories. It has been shown in [11] that stability is not invariant under bisimulation. Cuijpers shows that there exist two bisimilar systems such that one is Lyapunov stable with respect to a set of equilibrium points (trajectories which always remain at the same point), while the other is not. Hence, new notions of bisimulations with additional continuity requirements are introduced and shown to preserve Lyapunov stability with respect to a set of equilibrium points. More precisely, Cuijpers' result is as follows. Recall, a set of points X is Lyapunov stable if for every open neighborhood U of X, there exists a neighborhood V of X such that all trajectories starting from V remain within U. It is shown that if R is a simulation with certain continuity conditions on R and R^{-1}, then stability with respect to a set of points is preserved (See Theorem 2 of [11]).

We investigate a more general setting than that of Cuijpers, namely, we consider stability with respect to a set of executions, instead of a set of equilibrium points. We observe that the notion of continuous bisimulation introduced in [11] does not suffice when one considers stability of executions. In fact, it does not suffice even to reason about asymptotic stability with respect to a set of points. Next, we discuss these observations with counter-examples.

A. Insufficiency results

We begin by showing that the notion of continuous bisimulation introduced in [11] is not sufficient to preserve either Lyapunov or asymptotic stability of trajectories.

1) Lyapunov stability with respect to a set of executions: Consider two dimensional dynamical systems D_1 and D_2 whose phase portraits are shown in Figure 3 and Figure 4, respectively. The system evolves along the trajectories at a constant rate of 1. The reference set of executions in both cases is a singleton set consisting of the trajectory which evolves along the x-axis. Note that system D_1 is Lyapunov stable, since the trajectories move parallel to the x-axis, where as the system D_2 is unstable since the trajectories diverge. However, the bijection between the state-spaces of the two systems obtained by mapping the states on a trajectory of D_1 to the states on the trajectory in D_2 with the same initial state, is a bisimulation. Below we present the formal definitions of the systems.

The hybrid transition system corresponding to D_1 is $\mathcal{H}_1 = (S_1, \Sigma_1, \Delta_1)$, where

- the state-space S_1 is the set $\mathbb{R}^2_{\geq 0}$, which is the positive quadrant of the two dimensional plane;
- the set of transitions Σ_1 is the empty set; and
- Δ_1 is the set $\{f_m \,|\, m \in \mathbb{R}_{\geq 0}\}$, where for a particular $m \in \mathbb{R}_{\geq 0}$, $f_m : [0, \infty) \to \mathbb{R}^2_{\geq 0}$ is the trajectory such that $f_m(t) = (t, m)$.

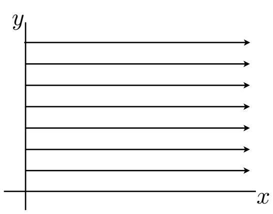


Fig. 3: D_1: A Lyapunov stable hybrid transition system. Trajectories are given by $((0,m),t) \mapsto (t,m)$. Reference trajectory starts at $(0,0)$.

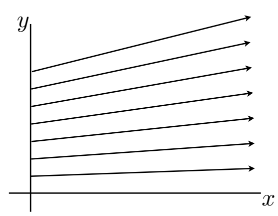


Fig. 4: D_2: An unstable hybrid transition system. Trajectories are given by $((0,m),t) \mapsto (t, m + mt)$. Reference trajectory starts at $(0,0)$.

The set of reference trajectories is given by $\mathcal{T}_1 = \{f_0\}$. Given any ϵ, one can choose $\delta = \epsilon$ such that *Lyap*$(\mathcal{H}_1, \mathcal{T}_1, \delta, \epsilon)$ holds. Hence, $\mathcal{H}_1$ is Lyapunov stable with respect to $\mathcal{T}_1$.

The hybrid transition system corresponding to D_2 is given by $\mathcal{H}_2 = (S_2, \Sigma_2, \Delta_2)$, where $S_2 = S_1$ and $\Sigma_2 = \Sigma_1$, and $\Delta_2 = \{f_m : [0, \infty) \to \mathbb{R}_{\geq 0} \,|\, f_m(t) = (t, m(1+t)), m \in \mathbb{R}_{\geq 0}\}$. The set of reference trajectories, $\mathcal{T}_2$, is the same as $\mathcal{T}_1$. Given any ϵ, for any $\delta > 0$, one can choose $T > \epsilon/\delta - 1$ such that the trajectory f_δ at times $t > T$ is at distance $> \epsilon$ from $\mathcal{T}_1$.

However, the relation $R = \{((x_1, y_1), (x_2, y_2)) \,|\, x_1 = x_2$ and $y_2 = y_1(1 + x_1)\}$ is a bisimulation relation between $\mathcal{H}_1$ and $\mathcal{H}_2$. Moreover, R is bi-continuous, that is, both R and R^{-1} (when interpreted as single valued functions) are continuous. This shows that bisimulation even with the additional continuity properties that subsume the requirements imposed in [11], does not suffice to preserve Lyapunov stability.

2) Asymptotic stability with respect to a set of executions: Let us consider a dynamical system D_3, which is similar to D_2, except that instead of the trajectories diverging from the x-axis, they converge exponentially to it. It is given by the hybrid transition system, $\mathcal{H}_3$, which is similar to $\mathcal{H}_1$ except that f_m is defined as $f_m(t) = (t, me^{-t})$. The set of reference trajectories is given by $\mathcal{T}_3 = \mathcal{T}_1$. The binary relation R' between $\mathcal{H}_1$ and $\mathcal{H}_3$, given by, $\{((x_1, y_1), (x_2, y_2)) \,|\; x_1 = x_2$ and $y_2 = y_1 e^{-x_1}\}$ is a bi-continuous bisimulation. R' essentially maps a trajectory in $\mathcal{H}_1$ to a trajectory in $\mathcal{H}_3$ with the same initial state. Note that $\mathcal{H}_1$ is not asymptotically stable, however, $\mathcal{H}_3$ is. So the continuity conditions in [11] on bisimulation relations, do not suffice to reason about asymptotic stability of trajectories. In fact, they do not suffice even to reason about asymptotic stability with respect to a set of points as shown below.

3) Insufficiency of the continuity conditions in [11] for asymptotic stability with respect to a set of equilibrium points: Consider a system $\mathcal{T}_1$ with state-space $\mathbb{R}$ such that every point in the system is an equilibrium point. More precisely, the trajectories are essentially constant function which always remain at the same point. Observe that $\mathcal{T}_2$ is stable with respect to equilibrium point 0, but not asymptotically stable.

Consider a system $\mathcal{T}_2$ with state-space $\mathbb{R}$ and the equilibrium point 0. Given any x, there is a trajectory starting at x which converges to 0 without crossing to the other side of 0. More precisely:

- If $x(0) > 0$, then $x(t) > 0$ for all t, $x(t_1) > x(t_2)$ for all $t_1 < t_2$, and $x(t) \to 0$ as $t \to \infty$.
- Similarly, if $x(0) < 0$, then $x(t) < 0$ for all t, $x(t_1) < x(t_2)$ for all $t_1 < t_2$, and $x(t) \to 0$ as $t \to \infty$.

Note that $\mathcal{T}_2$ is asymptotically stable with respect to the equilibrium point 0.

Now, we define a relation R from $\mathcal{T}_1$ to $\mathcal{T}_2$ satisfying the hypothesis of Theorem 2 of [11]: $R = \{(x, y) \in \mathbb{R}^2 \,|\, 0 < y \leq x$ or $x \leq y < 0\}$. R satisfies the following:

- R is a simulation.
- R^{-1} is upper semi-continuous.
- R is lower semi-continuous.

Hence, R satisfies the hypothesis of Theorem 2 in [11]. However, the conclusion of the theorem does not hold for asymptotic stability, because it would state that if $\mathcal{T}_2$ is asymptotically stable with respect to a closed set S, then $\mathcal{T}_1$ is asymptotically stable with respect to $R^{-1}(S)$, which does not hold since $\mathcal{T}_2$ is asymptotically stable with respect to $\{0\}$, however $\mathcal{T}_1$ is not asymptotically stable with respect to $R^{-1}(\{0\}) = \{0\}$.

B. Uniformly Continuous Simulations and Bisimulations

In this section, we introduce the notion of uniformly continuous simulations which add certain uniformity conditions on the relation, and show that they suffice to preserve both Lyapunov and asymptotic stability of trajectories.

Definition 13. A *uniformly continuous simulation* from an HTS $\mathcal{H}_1$ to an HTS $\mathcal{H}_2$ is a binary relation $R \subseteq S_1 \times S_2$ such that R is a simulation from $\mathcal{H}_1$ to $\mathcal{H}_2$, and R and R^{-1} are uniformly continuous functions.

The main result of this section is that uniformly continuous simulations serve as the right foundation for abstractions when verifying stability properties. That is, if $\mathcal{H}_1$ is uniformly

Proof.

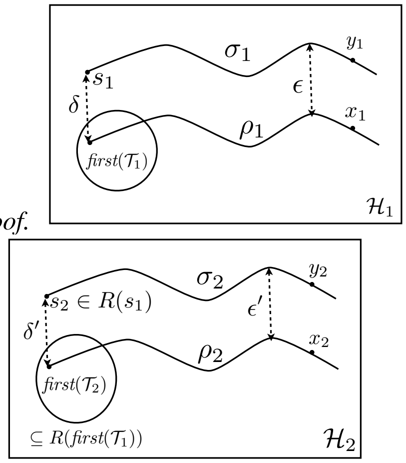


Fig. 5: Proof illustration

continuously simulated by $\mathcal{H}_2$ and $\mathcal{H}_2$ is stable with respect to $\mathcal{T}_2$, then $\mathcal{H}_1$ will be stable with respect to $\mathcal{T}_1$. However, for such an observation to hold, the simulation relation between $\mathcal{H}_1$ and $\mathcal{H}_2$ should also relate the executions of $\mathcal{T}_1$ and $\mathcal{T}_2$. So, before proving the main result of this section, we first formally define how the simulation relation should relate the sets $\mathcal{T}_1$ and $\mathcal{T}_2$.

Definition 14. Given HTSs $\mathcal{H}_1$ and $\mathcal{H}_2$, and sets of executions $\mathcal{T}_1 \subseteq Exec(\mathcal{H}_1)$ and $\mathcal{T}_2 \subseteq Exec(\mathcal{H}_2)$, a binary relation $R \subseteq S_1 \times S_2$ is said to be *semi-complete* with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$, if the following hold:

- C1 $R(First(\mathcal{T}_1)) \subseteq First(\mathcal{T}_2)$.
- C2 For every $\rho_2 \in \mathcal{T}_2$, there is an execution $\rho_1 \in \mathcal{T}_1$ such that $\rho_2 \in R(\rho_1)$.
- C3 For every $x \in States(\mathcal{T}_2)$, $R^{-1}(x)$ is a singleton.
- C4 There exists $\delta > 0$ such that for all $x \in B_\delta(First(\mathcal{T}_1))$, there exists a y such that $R(x, y)$.

R is *complete* with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$ if R and R^{-1} are semi-complete with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$.

The next theorem states that uniformly continuous simulations preserve Lyapunov and asymptotic stability.

Theorem 1 (Stability Preservation Theorem)**.** *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two hybrid transition systems and $\mathcal{T}_1 \subseteq Exec(\mathcal{H}_1)$ and $\mathcal{T}_2 \subseteq Exec(\mathcal{H}_2)$ be two sets of execution. Let $R \subseteq S_1 \times S_2$ be a uniformly continuous simulation from $\mathcal{H}_1$ to $\mathcal{H}_2$, and let R be semi-complete with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$. Then the following hold:*

1) *If $\mathcal{H}_2$ is Lyapunov stable with respect to $\mathcal{T}_2$, then $\mathcal{H}_1$ is Lyapunov stable with respect to $\mathcal{T}_1$.*
2) *If $\mathcal{H}_2$ is asymptotically stable with respect to $\mathcal{T}_2$, then $\mathcal{H}_1$ is asymptotically stable with respect to $\mathcal{T}_1$.*

(Lyapunov stability preservation) Let $\mathcal{H}_2$ be Lyapunov stable with respect to $\mathcal{T}_2$. We need to show that $\mathcal{H}_1$ is Lyapunov stable. Let us fix an $\epsilon > 0$. We need to find a δ such that $Lyap(\mathcal{H}_1, \mathcal{T}_1, \delta, \epsilon)$ holds. Choose ϵ' such that $UC(R^{-1}, \epsilon', \epsilon)$ holds, choose δ' such that $Lyap(\mathcal{H}_2, \mathcal{T}_2, \delta', \epsilon')$ holds and choose δ such that $UC(R, \delta, \delta')$ holds. Further, we can assume that δ is chosen small enough such that it satisfies Condition C4 of semi-completeness of R.

Next we show that $Lyap(\mathcal{H}_1, \mathcal{T}_1, \delta, \epsilon)$ for the δ chosen as above. Let σ_1 be an execution of $\mathcal{H}_1$ starting within a δ-ball around the initial states of $\mathcal{T}_1$, that is, $First(\sigma_1) \in B_\delta(First(\mathcal{T}_1))$. Let $First(\sigma_1) = s_1$. We need to find a $\rho_1 \in \mathcal{T}_1$ such that $d(\sigma_1, \rho_1) \leq \epsilon$. Since $s_1 \in B_\delta(First(\mathcal{T}_1))$, we have from Condition C4 of semi-completeness of R that there exists s_2 such that $R(s_1, s_2)$. Since R is a simulation, there exists a trajectory σ_2 from s_2 simulating σ_1. We claim that $s_2 \in B_{\delta'}(First(\mathcal{T}_2))$. Since $s_1 \in B_\delta(First(\mathcal{T}_1))$, there exists $s_1' \in First(\mathcal{T}_1)$ such that $d(s_1, s_1') \leq \delta$. Hence, every element $r \in R(s_1)$ is contained in $B_{\delta'}(R(s_1'))$ (since $UC(R, \delta, \delta')$ holds). In particular, $s_2 \in R(s_1)$ and hence $s_2 \in B_{\delta'}(R(s_1'))$. From Condition C1, since $s_1' \in First(\mathcal{T}_1)$, $R(s_1') \subseteq First(\mathcal{T}_2)$, therefore, $s_2 \in B_{\delta'}(First(\mathcal{T}_2))$. Hence, from Lyapunov stability of $\mathcal{H}_2$, there exists $\rho_2 \in \mathcal{T}_2$, $d(\sigma_2, \rho_2) \leq \epsilon'$. From Condition C2 of semi-completeness of R, there exists $\rho_1 \in \mathcal{T}_1$ such that $\rho_2 \in R(\rho_1)$.

We show that $d(\sigma_1, \rho_1) \leq \epsilon$. Let (t, i, y_1) be in $Gph(\sigma_1)$. Then there exists (t, i, y_2) in $Gph(\sigma_2)$ for some $y_2 \in R(y_1)$. There exists (t', i', x_2) in $Gph(\rho_2)$ within distance ϵ' from some (t, i, y_2). Also, there exists x_1 such that (t', i', x_1) in $Gph(\rho_1)$ and $x_1 \in R^{-1}(x_2)$. Since $d((t', i', x_2), (t, i, y_2)) \leq \epsilon'$, we have $|t - t'| \leq \epsilon'$, $|i - i'| \leq \epsilon'$ and $d(x_2, y_2) \leq \epsilon'$. Hence $R^{-1}(y_2) \subseteq B_\epsilon(R^{-1}(x_2))$, from the choice of ϵ'. From Condition C3, since $x_2 \in States(\mathcal{T}_2)$, $R^{-1}(x_2)$ is singleton and equal to x_1. Therefore $R^{-1}(y_2) \subseteq B_\epsilon(x_1)$. Since $y_1 \in R^{-1}(y_2)$, we have $d(x_1, y_1) \leq \epsilon$. Without loss of generality, we can assume that $\epsilon' \leq \epsilon$, hence, $|i - i'| \leq \epsilon$ and $|t - t'| \leq \epsilon$. Therefore, there exists $(t', i', x_1) \in Gph(\rho_1)$ within distance ϵ from (t, i, y_1). Given a $(t', i', x_1) \in Gph(\rho_1)$, we can find a $(t, i, y_1) \in Gph(\sigma_1)$ within distance ϵ from it in the same way.

(Asymptotic stability preservation) Let $\mathcal{H}_2$ be asymptotically stable with respect to $\mathcal{T}_2$. We will show that $\mathcal{H}_1$ is asymptotically stable with respect to $\mathcal{T}_1$. We know that $\mathcal{H}_1$ is Lyapunov stable from the first part. Let $\delta_2 > 0$ be such that for every σ_2 starting within a δ_2 ball of $First(\mathcal{T}_2)$, there exists a ρ_2 in $\mathcal{T}_2$, such that $Conv(\sigma_2, \rho_2)$. Let δ_1 be such that $UC(R, \delta_1, \delta_2)$ and small enough so that Condition C4 holds. Let σ_1 start in a $B_{\delta_1}(First(\mathcal{T}_1))$. We need to show that there exists a ρ_1 in $\mathcal{T}_1$ satisfying $Conv(\sigma_1, \rho_1)$. Similar to the argument in the previous part, there exists a σ_2 in $\mathcal{H}_2$ which simulates σ_1 and $d(First(\sigma_2), First(\mathcal{T}_2)) < \delta_2$. From the asymptotic stability of $\mathcal{H}_2$, there exists a ρ_2 in $\mathcal{T}_2$ such that $Conv(\sigma_2, \rho_2)$. Let ρ_1 be a trajectory in $\mathcal{T}_1$ such that $\rho_2 \in R(\rho_1)$ given by Condition C2 of semi-completeness of R. We will show that $Conv(\sigma_1, \rho_1)$. Let us fix an $\epsilon_1 > 0$. We need to show that there exist a $T \in \mathbb{R}_{\geq 0}$ such that $d(Gph(\sigma_1)|_T, Gph(\rho_1)|_T) < \epsilon_1$. Let us choose ϵ_2 such that $UC(R^{-1}, \epsilon_2, \epsilon_1)$ holds. There exist $T \in \mathbb{R}_{\geq 0}$ such that $d(Gph(\sigma_2)|_T, Gph(\rho_2)|_T) < \epsilon_2$. We can show that the same T works in $\mathcal{H}_1$ for the ϵ, that is, $d(Gph(\sigma_1)|_T, Gph(\rho_2)|_T) < \epsilon_1$. The proof is similar to showing that $d(\sigma_1, \rho_1) < \epsilon$ in the previous part (replace $Gph(\eta_i)$ by $Gph(\eta_i)|_T$). □

The above theorem essentially states that the stability of a system $\mathcal{H}_1$ can be inferred by analyzing a potentially simpler system $\mathcal{H}_2$ which uniformly continuously simulates $\mathcal{H}_1$.

Remark 4. We note here that stability preservation by our definition of simulation is not very tightly bound to the specific definition of distance between executions. For instance, our results carry over to the definitions of distance on the executions of the hybrid transition system, such as, Skorokhod metrics or the generalized Skorokhod metrics, discussed in Remark V-D. Simulation maps executions of a system to executions of another system with the same "hybrid time domain". Hence, the retimings in the time domains in the computation of the distance between executions of the same system, do not have an effect.

As a corollary of Theorem 1, we obtain that Lyapunov stability and asymptotic stability are invariant under uniformly continuous bisimulations.

Definition 15. A *uniformly continuous bisimulation* between two HTSs $\mathcal{H}_1$ and $\mathcal{H}_2$ is a binary relation $R \subseteq S_1 \times S_2$ such that R is a uniformly continuous simulation from $\mathcal{H}_1$ to $\mathcal{H}_2$ and R^{-1} is a uniformly continuous simulation from $\mathcal{H}_2$ to $\mathcal{H}_1$.

Theorem 2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two hybrid transition systems and $\mathcal{T}_1 \subseteq Exec(\mathcal{H}_1)$ and $\mathcal{T}_2 \subseteq Exec(\mathcal{H}_2)$ be two sets of execution. Let $R \subseteq S_1 \times S_2$ be a uniformly continuous bisimulation between $\mathcal{H}_1$ and $\mathcal{H}_2$, and let R be complete with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$. Then the following hold:*

1) *$\mathcal{H}_1$ is Lyapunov stable with respect to $\mathcal{T}_1$ if and only if $\mathcal{H}_2$ is Lyapunov stable with respect to $\mathcal{T}_2$.*
2) *$\mathcal{H}_1$ is asymptotically stable with respect to $\mathcal{T}_1$ if and only if $\mathcal{H}_2$ is asymptotically stable with respect to $\mathcal{T}_2$.*

Proof. R is a uniformly continuous simulation from $\mathcal{H}_1$ to $\mathcal{H}_2$ and is semi-complete with respect to $\mathcal{T}_1$ and $\mathcal{T}_2$. Therefore, from Theorem 1 if $\mathcal{H}_2$ is Lyapunov or asymptotically stable, then $\mathcal{H}_1$ is Lyapunov or asymptotically stable, respectively. Similarly, since R^{-1} is a uniformly continuous simulation from $\mathcal{H}_2$ to $\mathcal{H}_1$ and is semi-complete with respect to $\mathcal{T}_2$ and $\mathcal{T}_1$, we obtain that if $\mathcal{H}_1$ is Lyapunov or asymptotically stable, then $\mathcal{H}_2$ is Lyapunov or asymptotically stable, respectively. □

The above theorem suggests that uniformly continuous bisimulations can be interpreted as a notion of equivalence between systems when reasoning about stability.

1) Pre-orders on hybrid transition systems: We can define an ordering between hybrid transition systems using the notions of uniformly continuous simulations and bisimulations. We define a binary relation $\preceq_{UC}$ on the set of hybrid transition systems as follows. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be hybrid transition systems.

$\mathcal{H}_1 \preceq_{UC} \mathcal{H}_2$ if there exists a uniformly continuous simulation relation R from $\mathcal{H}_1$ to $\mathcal{H}_2$.

We show that $\preceq_{UC}$ is a pre-order on the class of hybrid transition systems, that is, a reflexive, transitive relation.

Corollary 1. *Uniformly continuous relations define a pre-order on the class of hybrid transition systems.*

Proof. Reflexivity: $\mathcal{H} \preceq_{UC} \mathcal{H}$, since the identity relation is a uniformly continuous simulation from $\mathcal{H}$ to itself.

Transitivity: Suppose $\mathcal{H}_1 \preceq_{UC} \mathcal{H}_2$ and $\mathcal{H}_2 \preceq_{UC} \mathcal{H}_3$. Let R_1 be a uniformly continuous simulation from $\mathcal{H}_1$ to $\mathcal{H}_2$ and R_2 be a uniformly continuous simulation from $\mathcal{H}_2$ to $\mathcal{H}_3$. Firstly, $R = R_2 \circ R_1$ is a simulation from $\mathcal{H}_1$ to $\mathcal{H}_3$, since simulations compose. Further, since R_1, $(R_1)^{-1}$, R_2 and $(R_2)^{-1}$ are uniformly continuous functions, from Proposition 2, we obtain that $(R_2 \circ R_1)$ and $(R_1)^{-1} \circ (R_2)^{-1}$ are uniformly continuous functions. But $(R_1)^{-1} \circ (R_2)^{-1} = (R_2 \circ R_1)^{-1}$. Hence, R and R^{-1} are uniformly continuous functions. Therefore, $\mathcal{H}_1 \preceq_{UC} \mathcal{H}_3$. □

VIII. Applications of the Stability Preservation Theorem

In this section, we demonstrate that uniformly continuous simulations and bisimulations have the potential to serve as the basis of simplification based stability analysis. The broad goal of these definitions is to serve as the guiding principles for developing new abstraction/minimization based stability analysis techniques. Here, we show that classical techniques for proving Lyapunov and asymptotic stability of systems can be formulated as constructing a simpler system which uniformly continuously simulates the original system and showing that the simpler system is Lyapunov or asymptotically stable, respectively. In addition, we also present a new abstraction based stability analysis technique motivated by the results in this paper.

A. Lyapunov Functions

We will show that Lyapunov's direct method for proving stability of dynamical systems can be interpreted as constructing a simpler one dimensional system which uniformly continuously simulates the original system using the Lyapunov function which serves as the uniformly continuous simulation, and then establishing the stability of the simpler system. Thus, Theorem 1 gives us the stability of the original system.

Consider the following time-invariant system,

$$\dot{x} = f(x), \qquad x(0) \in \mathbb{R}^n, \tag{3}$$

where $f : \mathbb{R}^n \to \mathbb{R}$. Let $\bar{0}$ be an equilibrium point of the system, that is, $f(\bar{0}) = \bar{0}$.

We associate a hybrid transition system $\mathcal{H}_f = (S, \Sigma, \Delta)$ with the dynamical system in (3), where $S = \mathbb{R}^n$, $\Sigma = \emptyset$, Δ is the set of C^1 trajectories[1] $\tau : I \to \mathbb{R}^n$ (where $I \in Int$) such that $d\tau(t)/dt = f(\tau(t))$. Let the metric d on the state-space S be the Euclidean distance. Let $\mathcal{T}_{f,x}$ be the set of all trajectories $\tau \in \Delta$ corresponding to an equilibrium point x, that is, τ such that $\tau(t) = x$ for all $t \in Dom(\tau)$.

Next we state Lyapunov's theorem which provides a sufficient condition for the stability of a system.

Theorem 3 (Lyapunov [29]). *Suppose that there exists a neighborhood Ω of $\bar{0}$ and a positive definite C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying the algebraic condition:*

$$\dot{V}(x) \leq 0, \qquad \forall x \in \Omega, \tag{4}$$

where $\dot{V}(x) = \frac{\partial V}{\partial x} f(x)$. Then System (3) is Lyapunov stable.

[1] C^1 is the set of continuously differentiable functions.

Furthermore, if $\dot{V}$ satisfies

$$\dot{V}(x) < 0, \qquad \forall x \in \Omega/\{0\}, \tag{5}$$

then System (3) is asymptotically stable.

A C^1 positive definite function satisfying inequality (4) is called a *weak Lyapunov function* for f over Ω, and one satisfying (5) is called a *Lyapunov function*. Let us say that a function $F : \mathbb{R}^n \to \mathbb{R}$ has *non-zero differential* if there exists a neighborhood U containing $\bar{0}$ such that the gradient of F at any point $x \in U$ other than $\bar{0}$, $\nabla F(x)$, is non-zero. Note that a Lyapunov function always has non-zero differential, whereas, a weak Lyapunov function may or may not have a non-zero differential.

The following theorem formulates Lyapunov's first method as a stability preserving reduction to a simpler system using uniformly continuous simulations.

Theorem 4. *Let $\dot{x} = f(x)$, $x(0) \in \mathbb{R}^n$ be a dynamical system with an equilibrium point $\bar{0}$. Suppose that $V : \mathbb{R}^n \to \mathbb{R}$ is a (non-zero differential weak) Lyapunov function for the dynamical system. Then, there exists Ω', a compact subset of Ω containing an open neighborhood of $\bar{0}$, such that the function $V_{\Omega'}$ (the restriction of the domain of V to Ω') satisfies:*

- *$V_{\Omega'}(\mathcal{H}_f)$ is (Lyapunov) asymptotically stable with respect to $V_{\Omega'}(\mathcal{T}_{f,\bar{0}})$.*
- *$V_{\Omega'}$ is a uniformly continuous simulation which is semi-complete with respect to $\mathcal{T}_{f,\bar{0}}$ and $V_{\Omega'}(\mathcal{T}_{f,\bar{0}})$.*

Therefore, $\mathcal{H}_f$ is (Lyapunov) asymptotically stable.

So, Lyapunov functions provide a concrete method for proving stability using our framework, namely, they serve as an abstraction function, application of which on a concrete system results in a simpler one dimensional abstract system which is stable, thereby implying the stability of the original system.

Remark 5. Note that we impose the constraint of non-zero differential in an open neighborhood around the equilibrium point on the weak Lyapunov function. However, this is not a serious restriction since a large fraction of candidate Lyapunov functions including the class of polynomial functions satisfy this requirement.

B. Multiple Lyapunov Functions

We show that proving stability of switched systems using multiple Lyapunov functions can be recast into the framework of Theorem 1.

A *switched system* consists of a set of dynamical systems and a switching signal which specifies the times at which the system switches its dynamics. Let us fix the following switched system with N dynamical systems.

$$\dot{x} = f_p(x), \; p \in [N], \; x(0) \in \mathbb{R}^n,$$
$$\alpha = (\{t_i\}_{i \in \mathbb{N}}, \{\omega_i\}_{i \in \mathbb{N}}), t_i \in \mathbb{R}_{\geq 0}, \omega_i \in [N]. \tag{6}$$

The *switching signal* $\alpha = (\{t_i\}_{i \in \mathbb{N}}, \{\omega_i\}_{i \in \mathbb{N}})$ is a monotonically increasing divergent sequence, that is, it satisfies $t_0 = 0$, $t_i < t_j$ for $j > i$ and for every $T \in \mathbb{R}_{\geq 0}$, there exists a k such that $t_k > T$.

The solution of this system is the set of functions $\sigma : [0, \infty) \to \mathbb{R}^n$ such that σ restricted to the interval between two switching times is a solution to the corresponding differential equation. Let $\sigma[a, b]$ denote the function from $[0, b-a]$ to $\mathbb{R}^n$ such that $\sigma[a, b](t) = \sigma(a + t)$. σ is a solution of (6) if for every $i \in \mathbb{N}$, $\sigma[t_i, t_{i+1}]$ is a solution of the differential equation $\dot{x} = f_{\omega_i}(x)$.

We associate an HTS $\mathcal{H}_{f_1,\ldots,f_N,\alpha}$ with the switched system in (6) given by (S, Σ, Δ), where,

- $S = \mathbb{R}_{\geq 0} \times \mathbb{R}^n$,
- $\Sigma = \emptyset$, and
- Δ consists of trajectories τ such that there exists an $i \in \mathbb{N}$ and a trajectory $\theta : [0, t_{i+1} - t_i] \to \mathbb{R}^n$ which is a solution of the differential equation $\dot{x} = f_{\omega_i}(x)$ such that $Dom(\tau) = Dom(\theta)$ and $\tau(t) = (t_i + t, \theta(t))$.

In an execution, the first component time is used to identify the mode of the system, that is, the dynamics associated with the trajectory. Hence, we define the metric d_t over a space $\mathbb{R}_{\geq 0} \times \mathbb{R}^n$ as, $d_t((t_1, x_1), (t_2, x_2))$ is Euclidean distance between x_1 and y_1 if $t_1 = t_2$, and ∞ otherwise. The metric space associated with S will be d_t. Let $\mathcal{T}_{f_1,\ldots,f_N,\alpha,\bar{0}}$ be the singleton set with the execution $\sigma : \mathbb{N} \to \Delta$, such that for each i and $t \in [0, t_{i+1} - t_i]$, $\sigma(i)(t) = (t_i + t, \bar{0})$.

Next, we state a result on the multiple Lyapunov method for stability analysis. Given a switching sequence $\alpha = (\{t_i\}_{i \in \mathbb{N}}, \{\omega_i\}_{i \in \mathbb{N}})$, we say that t_i and t_j are *adjacent* if t_j is the first switching time after t_i such that $\omega_i = \omega_j$.

Theorem 5 (Multiple Lyapunov Method [6])**.** *Let us consider the switched system of (6). Suppose there exist N weak Lyapunov functions $V_1, \ldots, V_N$ for $f_1, \ldots, f_N$, respectively, over a neighborhood Ω of $\bar{0}$ such that for any pair of adjacent switching times t_i and t_j, $V_{\omega_i}(\sigma(t_j)) \leq V_{\omega_i}(\sigma(t_i))$, for every solution σ of the switched system. Then the switched system is Lyapunov stable.*

We call a vector of functions $\bar{V} = (V_1, \ldots, V_N)$ satisfying the hypothesis of Theorem 5, a *multiple weak Lyapunov function* for the switched system (6).

The above theorem can again be formulated as establishing a function from the HTS $\mathcal{H}_{f_1,\ldots,f_N,\alpha}$ to a simpler HTS using the functions $V_1, \ldots, V_N$ such that the simpler system is Lyapunov stable and the mapping is a uniformly continuous simulation, thereby proving the stability of the original system.

Given a vector of functions $\bar{F} = (F_1, \ldots, F_k)$, where $F_i : \mathbb{R}^m \to \mathbb{R}$ for $1 \leq i \leq k$, and a switching signal $\beta = (\{t_i\}_{i \in \mathbb{N}}, \{\omega_i\}_{i \in \mathbb{N}})$, we define a function $\bar{F}[\beta] : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \to \mathbb{R}_{\geq 0} \times \mathbb{R}$, such that $\bar{F}[\beta](t, x) = (t, F_{\omega_i}(x))$, where $t \in [t_i, t_{i+1}]$.

Theorem 6. *Given the switched system in Equation (6), let $\bar{V}$ be a multiple weak Lyapunov function for the switched system, such that for each $1 \leq i \leq N$, V_i has a non-zero differential. Then, there exists a neighborhood $\bar{\Omega}$ of $\mathbb{R}_{\geq 0} \times \bar{0}$ such that $\bar{V}[\alpha]_{\bar{\Omega}}$, the restriction of the domain of $\bar{V}[\alpha]$ to $\bar{\Omega}$, satisfies:*

- *$\bar{V}[\alpha]_{\bar{\Omega}}(\mathcal{H}_{f_1,\ldots,f_N,\alpha})$ is Lyapunov stable with respect to $\bar{V}[\alpha]_{\bar{\Omega}}(\mathcal{T}_{f_1,\ldots,f_N,\alpha,\bar{0}})$.*

- $\bar{V}[\alpha]_{\bar{\Omega}}$ *is a uniformly continuous simulation which is semi-complete with respect to the sets* $\mathcal{T}_{f_1,\ldots,f_N,\alpha,\bar{0}}$ *and* $\bar{V}[\alpha]_{\bar{\Omega}}(\mathcal{T}_{f_1,\ldots,f_N,\alpha,\bar{0}})$.

Therefore, $\mathcal{H}_{f_1,\ldots,f_N,\alpha}$ *is Lyapunov stable.*

The metric associated with $\mathcal{H}_{f_1,\ldots,f_N,\alpha}$ and $\bar{V}[\alpha]_{\bar{\Omega}}(\mathcal{H}_{f_1,\ldots,f_N,\alpha})$ is d_t.

C. Hartman-Grobman Theorem

We consider a theorem due to Hartman-Grobman which constructs linear approximations of non-linear dynamics and establishes a homeomorphism between the two dynamics. We show that the homeomorphic mapping from the non-linear dynamics to the linear dynamics is a uniformly continuous bisimulation. And hence one can use these reductions from non-linear to linear dynamics to potentially establish stability properties of non-linear dynamics by proving stability of the simpler linear dynamics, and using Theorem 1 to deduce the stability of the non-linear dynamics.

Theorem 7 (Hartman-Grobman Theorem)**.** *Consider a system* $\dot{x} = F(x)$*, where* $F : \mathbb{R}^n \to \mathbb{R}^n$ *is a continuously differentiable function. Suppose that* $x_0 \in \mathbb{R}^n$ *is a hyperbolic equilibrium point of the system, that is,* $A = DF(x_0)$ *is a hyperbolic matrix, where* DF *denotes the Jacobian of* F*. Let* φ *be the flow generated by the system, that is,* $\varphi : \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ *is a differentiable function such that* $d\varphi(x,t)/dt = F(\varphi(x,t))$ *for all* $t \in \mathbb{R}_{\geq 0}$.

Then there are neighborhoods U *and* V *of* x_0 *and a homeomorphism* $h : U \to V$ *such that* $\varphi(h(x),t) = h(x_0 + e^{tA}(x - x_0))$*, whenever* $x \in U$ *and* $x_0 + e^{tA}(x - x_0) \in V$.

Let us call a function h satisfying the above condition, a *Hartman-Grobman* function associated with the dynamical system $\dot{x} = F(x)$.

Remark 6. The terminologies referred to in the above theorem are standard. However, we define them in the Appendix for the sake of completeness.

Given an *HTS* $\mathcal{H} = (S, \Sigma, \Delta)$ and a set $X \subseteq S$, the restriction of $\mathcal{H}$ to X, denoted $\mathcal{H} \cap X$, is the *HTS* $(X, \Sigma \cap \mathit{Trans}(X), \Delta \cap \mathit{Traj}(X))$.

Theorem 8. *Let* $\dot{x} = F(x)$ *be a dynamical system, where* F *is continuously differentiable,* x_0 *a hyperbolic equilibrium point,* $A = DF(x_0)$ *and* h *a Hartman-Grobman function for* $\dot{x} = F(x)$*. Let* $\mathcal{H}_A$ *and* $\mathcal{H}_F$ *be the hybrid transition systems associated with* $\dot{x} = Ax$ *and* $\dot{x} = F(x)$*, respectively. Then, there exists a compact set* Ω *containing a neighborhood of* x_0 *such that* h_Ω*, the restriction of the domain of* h *to* Ω*, satisfies:*

- h_Ω *is a uniformly continuous bisimulation between* $\mathcal{H}_A \cap \Omega$ *and* $\mathcal{H}_F \cap \Omega$ *which is semi-complete with respect to* $\mathcal{T}_{A,x_0}$ *and* $\mathcal{T}_{F,x_0}$.

Therefore, $\mathcal{H}_F$ *is (Lyapunov) asymptotically stable if and only if* $\mathcal{H}_A$ *is (Lyapunov) asymptotically stable.*

The Hartman-Grobman function is a uniformly continuous bisimulation from a non-linear system to its linearization. Hence, it provides a state-space reduction mechanism to a simpler system, namely, linear system, from which stability of the non-linear system can be inferred. Note that unlike in the previous methods, the dimension of the abstract system is the same as that of the concrete system, nonetheless, it is simpler in terms of the complexity of verification. The stability of a linear dynamical system is characterized by the eigen values of the matrix representing it.

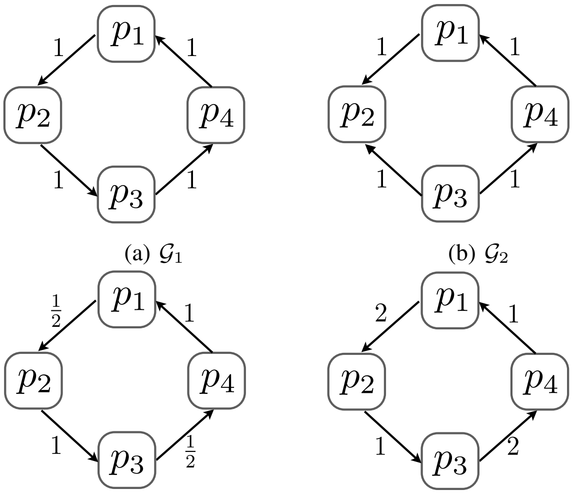
D. Predicate Abstraction for Stability Analysis

Next, we discuss a new abstraction based technique for stability analysis which is inspired by the results in this paper. It is an extension of the predicate abstraction technique discussed in Section II and can be found in [36]. Here, we briefly explain the abstraction technique and discuss its relation to the results in this paper.

The predicate abstraction technique in [36] extends the classical predicate abstraction by annotating the edges with weights which capture the evolution of the distance of the state to the equilibrium point. The weighted abstract graphs corresponding to the systems in Figure 1 are shown in Figure 6. The weight on an edge from p_i to p_j is an upper bound on the ratio of the distance to the origin of an execution when it reaches p_j to its distance to the origin when it started in p_i. For instance, in System 4, an execution starting on p_1 at distance 1 to the origin (the point $(0,1)$) will reach the state $(-2,0)$ on p_2 which is at distance 2 to the origin; the weight 2 on the edge from p_1 to p_2 is an upper bound on the ratio $2/1$ (and the scaling of all executions from p_1 to p_2). The system is Lyapunov stable if the graph does not contain cycles with product of weights on the edges > 1 (≥ 1 for asymptotic stability). Hence, we obtain from the weighted abstract graphs of Figure 6, that Systems 1 and 2 are Lyapunov stable, System 4 is asymptotically stable and System 3 is potentially unstable. For two dimensional systems of the dynamics considered here [38], the converse also holds, that is, if the system is Lyapunov stable, then there is a constructible weighted graph in which the product of the weights on the edges of any cycle is ≤ 1 (and < 1 for asymptotic stability).

Now, we briefly explain how this abstraction method can be interpreted as a uniformly continuous simulation from the concrete system to the abstract weighted graph. Details for a general class of systems can be found in [37]. We formally represent the systems in Figure 1 as hybrid transition systems with only trajectories. For instance, the hybrid transition system representing System k in Figure 1 is the tuple $\mathcal{H}_k = (\mathbb{R}^2, \emptyset, \Delta)$, where Δ consists of trajectories $\tau : \mathbb{R}_{\geq 0} \to \mathbb{R}^2$ such that there exists a sequence of diverging times $0 = t_0 < t_1 < t_2 < \ldots$ and a sequence of indices $i_0, i_1, i_2, \ldots$ such that $i_{j+1} = (i_j \mathit{mod}\, 4) + 1$, and τ restricted to the interval $[t_j, t_{j+1}]$ satisfies the differential equation corresponding to the quadrant between the lines p_{i_j} and $p_{i_{j+1}}$ (recall p_1 corresponds to the positive y-axis, p_2 to negative x-axis, p_3 to negative y-axis and p_4 to the positive x-axis).

A weighted abstract graph $\mathcal{G}_k$ for a system S_k can be interpreted as a one dimensional hybrid system $\mathcal{H}_{\mathcal{G}_k}$, which consists only of trajectories evolving along a path of the graph and respecting the weights on the edges. More precisely, the hybrid transition systems corresponding to the weighted



(a) $\mathcal{G}_1$ (b) $\mathcal{G}_2$ (c) $\mathcal{G}_3$ (d) $\mathcal{G}_4$

Fig. 6: Weighted Abstract Graphs of Systems 1, 2, 3 and 4 from Figure 1

graphs in Figure 6 are given by tuples $(\mathbb{R}_{\geq 0}, \emptyset, \hat{\Delta})$, where $\hat{\Delta}$ consists of trajectories $\hat{\tau} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that there exists a sequence of diverging times $0 = t_0 < t_1 < t_2 < \dots$ and a sequence of indices $i_0, i_1, i_2, \dots$ such that $i_{j+1} = (i_j \, mod \, 4) + 1$, and $\hat{\tau}$ restricted to the interval $[t_j, t_{j+1}]$, namely $\hat{\tau}[t_j, t_{j+1}]$, respects the the weight on the edge from p_{i_j} to $p_{i_{j+1}}$, that is, $\frac{\hat{\tau}[t_j,t_{j+1}](t')}{\hat{\tau}[t_j,t_{j+1}](0)} \leq$ the weight on the edge $(p_{i_j}, p_{i_{j+1}})$ for all $t' \in [t_j, t_{j+1}]$. This translation has the property that $\mathcal{H}_{G_k}$ is Lyapunov (asymptotically) stable if and only if the graph $\mathcal{G}_k$ does not have any cycles with product of weight > 1 (≥ 1).

Next, $\mathcal{H}_{\mathcal{G}_k}$ uniformly continuously simulates $\mathcal{H}_k$ using the relation $(\|x\|, x)$, where $x \in \mathbb{R}^2$ and $\|x\|$ is the Euclidean norm of x. Note that for any trajectory τ of $\mathcal{H}_k$, the trajectory $\hat{\tau}$, given by $\hat{\tau}(t) = \|\tau(t)\|$, is in the one dimensional hybrid system $\mathcal{H}_{\mathcal{G}_k}$.

In conclusion, the negative results in the paper on insufficiency of simulation and bisimulation suggest that novel abstraction mechanisms need to be devised for stability analysis. Theorem 1 and Theorem 2 provide the foundational principle for concrete abstraction and minimization techniques for stability analysis. The predicate abstraction technique from [36] is one such technique based on the foundations established in this paper.

IX. Conclusions

In this paper, we investigated pre-orders for reasoning about stability properties of dynamical and hybrid systems. We showed that bisimulation relations with continuity conditions, introduced in [11], are inadequate when stronger notions of stability like asymptotic stability, or the stability of trajectories is considered. We, therefore, introduced uniformly continuous simulations and bisimulations and showed that they form the semantic basis to reason about stability. Using such notions, we showed that, classical reasoning principles in control theory can be recast in a more “computer-science-like light”, wherein they can be seen as being founded on abstracting/simplifying a system and then relying on the reflection of certain logical properties by the abstraction relation.

As argued in [11], one by-product of investigating the continuity requirements on simulations and bisimulations needed to reason about stability, is that it allows one to conclude the inadequacy of the modal logic in [13], [12], [15] to express stability properties. What is the right logic to express properties like stability? That remains open. Just like Hennessy-Milner logic serves as the logical foundation for classical simulation and bisimulation, the right modal logic that can express stability might form the logical basis for the simulation and bisimulation relations introduced here.

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References

[1] R. Alur, T. Dang, and F. Ivancic. Counter-Example Guided Predicate Abstraction of Hybrid Systems. In *Tools and Algorithms for the Construction and Analysis of Systems*, pages 208–223, 2003.

[2] R. Alur, T. Dang, and F. Ivancic. Counter-Example Guided Predicate Abstraction of Hybrid Systems. In *Proceedings of the International Conference on Tools and Algorithms for the Construction and Analysis of Systems*, pages 208–223, 2003.

[3] R. Alur and D. Dill. A theory of timed automata. *Theoretical Computer Science*, 126:183–235, 1994.

[4] R. Alur, T. Henzinger, G. Lafferriere, George, and G. J. Pappas. Discrete abstractions of hybrid systems. In *Proceedings of the IEEE*, pages 971–984, 2000.

[5] E. Asarin, T. Dang, and A. Girard. Hybridization methods for the analysis of nonlinear systems. *Acta Informatica*, 43(7):451–476, 2007.

[6] M. S. Branicky. Stability of hybrid systems: state of the art. In *Conference on Decision and Control*, pages 120–125, 1997.

[7] T. Brihaye and C. Michaux. On the expressiveness and decidability of o-minimal hybrid systems. *Journal of Complexity*, 21(4):447–478, 2005.

[8] E. Clarke, A. Fehnker, Z. Han, B. Krogh, J. Ouaknine, O. Stursberg, and M. Theobald. Abstraction and Counterexample-Guided Refinement in Model Checking of Hybrid Systems. *International Journal on Foundations of Computer Science*, 14(4):583–604, 2003.

[9] E. Clarke, O. Grumberg, S. Jha, Y. Lu, and H. Veith. Counterexample-Guided Abstraction Refinement. In *Computer Aided Verification*, pages 154–169, 2000.

[10] E. M. Clarke, Jr., O. Grumberg, and D. A. Peled. *Model Checking*. MIT Press, Cambridge, MA, USA, 1999.

[11] P. J. L. Cuijpers. On bicontinuous bisimulation and the preservation of stability. In *Proceedings of the International Conference on Hybrid Systems: Computation and Control*, pages 676–679, 2007.

[12] J. Davoren. Topological semantics and bisimulations for intuitionistic modal logics and their classical companion logics. In *Proceedings of the International Conference on Logical Foundations of Computer Science*, pages 162–180, 2007.

[13] J. Davoren and R. Goré. Bimodal logics for reasoning about continuous dynamics. In *Advances in Modal Logic*, volume 3, pages 91–110. 2002.

[14] J. M. Davoren. Epsilon-tubes and generalized skorokhod metrics for hybrid paths spaces. In *Proceedings of the International Conference on Hybrid Systems: Computation and Control*, pages 135–149, 2009.

[15] G. Fainekos and G. Pappas. Robustness of tenporal logic specifications. In *Proceedings of the International Conference on Formal Approaches to Testing and Runtime Verification*, 2006.

[16] A. Geletu. *Introduction to Topological Spaces and Set-Valued Maps*. Institute of Mathematics, Department of Operations Research and Stochastics, Ilmenau University of Technology, 2006.

[17] A. Girard, A. A. Julius, and G. J. Pappas. Approximate simulation relations for hybrid systems. *Discrete Event Dynamic Systems*, 18(2):163–179, 2008.

[18] A. Girard and G. J. Pappas. Approximate bisimulation relations for constrained linear systems. *Automatica*, 43(8):1307–1317, 2007.
[19] A. Girard, G. Pola, and P. Tabuada. Approximately bisimilar symbolic models for incrementally stable switched systems. In *Proceedings of the International Conference on Hybrid Systems: Computation and Control*, pages 201–214, 2008.
[20] R. Goebel, R. Sanfelice, and A. Teel. Hybrid dynamical systems. *IEEE Control Systems, Control Systems Magazine*, 29:28–93, 2009.
[21] S. Graf and H. Saidi. Construction of abstact state graphs with PVS. In *Proceedings of the International Conference on Computer Aided Verification*, pages 72–83, 1997.
[22] D. Grobman. Homeomorphism of systems of differential equations. *Doklady Akad. Nauk SSSR*, 128:880–881, 1959.
[23] D. Grobman. Topological classification of neighborhoods of a singularity in n-space. *Mat. Sbornik*, 56(98):77–94, 1962.
[24] P. Hartman. A lemma in the theory of structural stability of differential equations. *Proceedings of the American Mathematical Society*, 11:610–620, 1960.
[25] T. A. Henzinger. The Theory of Hybrid Automata. In *Logic In Computer Science*, pages 278–292, 1996.
[26] T. A. Henzinger, P. W. Kopke, A. Puri, and P. Varaiya. What's decidable about hybrid automata? In *Proceedings of the Twenty-seventh Annual ACM Symposium on Theory of Computing*, STOC '95, pages 373–382, New York, NY, USA, 1995. ACM.
[27] H. F. Jean-Pierre Aubin. *Set-valued Analysis*. Boston : Birkhauser, 1990.
[28] D. K. Kaynar, N. A. Lynch, R. Segala, and F. W. Vaandrager. Timed I/O Automata: A Mathematical Framework for Modeling and Analyzing Real-Time Systems. In *Real-Time Systems Symposium*, pages 166–177. IEEE Computer Society, 2003.
[29] H. K. Khalil. *Nonlinear Systems*. Prentice-Hall, Upper Saddle River, NJ, 1996.
[30] G. Lafferriere, G. Pappas, and S. Sastry. O-minimal Hybrid Systems. *Mathematics of Control, Signals, and Systems*, 13(1):1–21, 2000.
[31] D. Lee and M. Yannakakis. Online Minimization of Transition Systems (Extended Abstract). In *Symposium on Theory Of Computing*, pages 264–274. ACM, 1992.
[32] D. Liberzon. *Switching in Systems and Control*. Birkhauser Basel, 2003.
[33] R. Milner. *Communication and Concurrency*. Prentice-Hall, Inc, 1989.
[34] P. Prabhakar, S. Duggirala, S. Mitra, and M. Viswanathan. Hybrid automata-based cegar for rectangular hybrid automata. In *Proceedings of the International Conference on Verification, Model Checking, and Abstract Interpretation*, 2013.
[35] P. Prabhakar, G. E. Dullerud, and M. Viswanathan. Pre-orders for reasoning about stability. In *Proceedings of the International Conference on Hybrid Systems: Computation and Control*, pages 197–206, 2012.
[36] P. Prabhakar and M. G. Soto. Abstraction based model-checking of stability of hybrid systems. In *CAV*, pages 280–295, 2013.
[37] P. Prabhakar and M. G. Soto. Foundations of quantitative predicate abstraction for stability analysis of hybrid systems. In *Proceedings of the International Conference on Verification, Model Checking, and Abstract Interpretation*, 2015.
[38] P. Prabhakar and M. Viswanathan. On the decidability of stability of hybrid systems. In *HSCC*, pages 53–62, 2013.
[39] P. Prabhakar, V. Vladimerou, M. Viswanathan, and G. Dullerud. Verifying tolerant systems using polynomial approximations. In *Proceedings of the IEEE Real Time Systems Symposium*, 2009.
[40] V. Vladimerou, P. Prabhakar, M. Viswanathan, and G. E. Dullerud. Stormed hybrid systems. In *ICALP (2)*, pages 136–147, 2008.

Appendix

A. Preliminaries on Set-Valued functions

First, we provide the proofs of Proposition 1 and Proposition 2, respectively.

Proposition *1:* If $F : A \rightsquigarrow B$ is a uniformly upper semi-continuous function, then F is also a lower semi-continuous function.

Proof. Let $V \subseteq B$ be an open set. We need to show that the set $U = \{a \in A \,|\, F(a) \cap V \neq \emptyset\}$ is open. Fix $a \in U$. It suffices to show that there exists a $\delta > 0$ such that $B_\delta(a) \subseteq U$. Equivalently, there exists a $\delta > 0$ such that for every a' such that $d(a, a') < \delta$, $a' \in U$.

Fix $b \in V$ such that $b \in F(a)$ (such a b exists since $F(a) \cap V \neq \emptyset$). Since V is a open set, there exists an $\epsilon > 0$ such that $B_\epsilon(b) \subseteq V$. Let $V' = B_\epsilon(b)$. Let $\delta > 0$ be such that $UC(F, \delta, \epsilon)$. We will show that for every a' such that $d(a, a') < \delta$, $F(a') \cap V' \neq \emptyset$, which implies $F(a') \cap V \neq \emptyset$ and hence $a' \in U$.

Fix a' such that $d(a, a') < \delta$. We need to show that $F(a') \cap V' \neq \emptyset$. Note that $F(a) \subseteq B_\epsilon(F(a'))$. Therefore, for every $b_1 \in F(a)$, there exists a $b_2 \in F(a')$ such that $d(b_1, b_2) < \epsilon$. This is equivalent to: for every $b_1 \in F(a)$, $B_\epsilon(b_1) \cap F(a') \neq \emptyset$. In particular, for $b_1 = b$, this implies that $V' \cap F(a') \neq \emptyset$. ☐

Proposition *2:* Let $F : A \rightsquigarrow B$ and $G : B \rightsquigarrow C$ be two set-valued uniformly continuous functions. Then: $G \circ F$ is a uniformly continuous function.

Proof. Proof of part (1): Given $\epsilon > 0$, there exists a γ such that $UC(G, \gamma, \epsilon)$. Similarly, there exists a δ such that $UC(F, \delta, \gamma)$. We claim that $UC(G \circ F, \delta, \epsilon)$. Let $a_1, a_2 \in A$ such that $d(a_1, a_2) < \delta$ and $c_1 \in (G \circ F)(a_1)$. We need to find a $c_2 \in (G \circ F)(a_2)$ such that $d(c_1, c_2) \leq \epsilon$. Since $c_1 \in (G \circ F)(a_1)$, there exists b_1 such that $b_1 \in G(a_1)$ and $c_1 \in F(b_2)$. From the uniform continuity of G, there exists b_2 such that $b_2 \in G(a_2)$ and $d(b_1, b_2) \leq \gamma$. Further, from the uniform continuity of F, there exists c_2 such that $c_2 \in F(b_2)$ and $d(c_1, c_2) \leq \epsilon$. But $c_2 \in (G \circ F)(a_2)$. Note that the above argument cannot be used to prove that composition of two continuous functions is continuous. The uniformity constraint allows us to choose a uniform γ for all elements in $F(a_1)$, corresponding to the ϵ. ☐

Next, we state some properties about upper semi-continuous and uniformly continuous functions, and a few additional concepts that will be useful in some of the proofs. A set-valued function $F : A \rightsquigarrow B$ is said to be closed valued (compact valued) if for every $a \in A$, the set $F(a)$ is closed (compact).

In general, the restriction of a upper semi-continuous set-valued map to a compact domain is not uniformly upper semi-continuous. However, for a slightly stronger notion of continuity, the proof of the above fact for single valued functions can be mimicked to obtain a proof for the set-valued function. Hence, we define, Hausdorff semi-continuity.

Definition 16. A set-valued function $F : A \rightsquigarrow B$ is said to be *Hausdorff lower semi-continuous* if

$$\forall \epsilon > 0, \forall a \in A, \exists \delta > 0, \forall a' \in B_\delta(a), F(a) \subseteq B_\epsilon(F(a')).$$

The definition of upper semi-continuity is similar to Hausdorff lower semi-continuity except for the requirement that $F(a') \subseteq B_\epsilon(F(a))$, instead of $F(a) \subseteq B_\epsilon(F(a))$. Note that for a single valued function, the two are the same. It is this asymmetry, which forbids the proof for the single-valued case to be carried over to the set-valued case.

We need a few additional definitions which we use in the proof of Proposition 3. For details, see the lecture notes [16]. A set-valued map $F : A \rightsquigarrow B$ is a closed map, if the relation F is closed set. Note that if F is closed, then so is F^{-1}.

Definition 17. A set-valued function $F : A \rightsquigarrow B$ is an *outer semi-continuous function* if for every $a_0 \in A$,

$$\limsup_{a \to a_0} F(a) \subseteq F(a_0),$$

where $\limsup_{x \to x^0} F(x) = \{y \mid \exists x_n \to x^0, \exists y_n \to y^0, y_n \in F(x_n)\}$.

Proposition 3. *Let $F : A \rightsquigarrow B$ be a set-valued function.*

- *If F is closed valued and upper semi-continuous and F^{-1} is closed valued, then F^{-1} is upper semi-continuous.*
- *If F is lower semi-continuous and compact valued, then F is Hausdorff lower semi-continuous.*

Proof. Proof of Part (1): From Corollary 5.3.26 of [16], F is closed valued and upper semi-continuous implies that F is a closed map. Then F^{-1} is a closed map. From Proposition 5.3.24 of [16], F^{-1} is outer semi-continuous. Finally from Proposition 5.3.29 of [16], since F^{-1} is closed valued and outer semi-continuous, F^{-1} is upper semi-continuous.

Proof of part (2): It is essentially Proposition 5.3.44 of [16]. □

Proposition 4. *If $F : A \rightsquigarrow B$ is upper semi-continuous and Hausdorff lower semi-continuous, and A is compact, then F is uniformly continuous.*

Proof. Proof is similar to that of the single valued case, that is, a single valued continuous function with a compact domain is uniformly continuous. □

B. Preliminaries for Section VIII

In this section, we recall certain standard definitions for dynamical systems.

Consider a C^1 (i.e., continuously differentiable) function $V : \mathbb{R}^n \to \mathbb{R}$. It is called *positive definite* if $V(\bar{0}) = 0$ and $V(x) > 0$ for all $x \neq 0$. Let

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x),$$

and note that $\dot{V}$ is the time derivative of $V(x(t))$, where $x(t)$ is a solution of the Equation 3.

Below, we present the definitions of terminologies used in Theorem 7. A function $f : A \to B$, where $A, B \subseteq \mathbb{R}^n$ is a *homeomorphism* if f is a bijection and both f and f^{-1} are continuous. A function $F : \mathbb{R}^n \to \mathbb{R}^m$ is given by m-real valued component functions, $y_1(x), \dots, y_m(x)$, where $x = (x_1, \dots, x_n)$. The partial derivatives of all these functions (if they exist) can be organized in a $m \times n$ matrix called the *Jacobian* of F, denoted by $DF(x)$, where the entry in the i-th row and j-th column is $\partial y_i / \partial x_j$.

Given an n-vector $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, $DF(a)$ is the matrix obtained by substituting x_i in the terms of the matrix $DF(x_1, \dots, x_n)$ by a_i. A square matrix A is *hyperbolic* if none of its eigen values are purely imaginary values (including 0).

C. Proof of Theorem 4

Theorem *4:* Let $\dot{x} = f(x)$, $x(0) \in \mathbb{R}^n$ be a dynamical system with an equilibrium point $\bar{0}$. Suppose that $V : \mathbb{R}^n \to \mathbb{R}$ is a (non-zero differential weak) Lyapunov function for the dynamical system. Then, there exists Ω', a compact subset of Ω containing an open neighborhood of $\bar{0}$, such that the function $V_{\Omega'}$ (the restriction of the domain of V to Ω') satisfies:

- $V_{\Omega'}(\mathcal{H}_f)$ is (Lyapunov) asymptotically stable with respect to $V_{\Omega'}(\mathcal{T}_{f,\bar{0}})$.
- $V_{\Omega'}$ is a uniformly continuous simulation which is semi-complete with respect to $\mathcal{T}_{f,\bar{0}}$ and $V_{\Omega'}(\mathcal{T}_{f,\bar{0}})$.

Therefore, $\mathcal{H}_f$ is (Lyapunov) asymptotically stable.

Proof. Let $\mathcal{H}_f = (S_1, \Sigma_1, \Delta_1)$. Choose Ω' to be a compact sub-level set of V around $\bar{0}$ containing an open neighborhood of $\bar{0}$ and $\Omega' \subseteq \Omega$ (where Ω is the neighborhood in the definition of Lyapunov function). Further, when V has non-zero differential, then Ω' is contained in the open neighborhood around $\bar{0}$ with no zero differentials. Let $V_{\Omega'}(\mathcal{H}_f) = (S_2, \Sigma_2, \Delta_2)$.

Let V be a non-zero differential weak Lyapunov function. First, we show that $V_{\Omega'}(\mathcal{H}_f)$ is Lyapunov stable. Let $\tau_2 \in \Delta_2$. Then, there exists a $\tau_1 \in \Delta_1$ such that $\tau_2 = V_{\Omega'}(\tau_1)$. The condition $\dot{V}(x) \leq 0$ implies that for any trajectory $\tau \in \Delta_1$, $d/dt V(\tau(t)) \leq 0$. In other words, the image under V of τ decreases. Therefore, the image under V of τ_1 decreases implying that the value of τ_2 decreases as time increases, that is, for any $t_1 \leq t_2$, $\tau_2(t_1) \geq \tau_2(t_2)$. Hence, given any $\epsilon > 0$, any trajectory starting within an ϵ-ball around $\bar{0}$ remains in the ϵ-ball in $V_{\Omega'}(\mathcal{H}_f)$.

Next, we show that $V_{\Omega'}$ is a uniformly continuous simulation. Note that since every trajectory starting in the domain of $V_{\Omega'}$ remains in the domain (since Ω' is a sub-level set), the image of the trajectory is in $V_{\Omega'}(\mathcal{H}_f)$. Therefore, $V_{\Omega'}$ is a simulation. $V_{\Omega'}$ is a uniformly continuous single valued function, since it is continuous and its domain is compact. Hence, it is also a uniformly upper semi-continuous function. Also, note that $V_{\Omega'}(\Omega')$, say Ω'', is compact by definition. We will show that $V_{\Omega'}^{-1} : \Omega'' \rightsquigarrow \Omega$ is uniformly upper semi-continuous. Note that $V_{\Omega'}^{-1}$ is closed valued, since level sets of a Lyapunov function are closed. Therefore, from Proposition 3, $V_{\Omega'}^{-1}$ is upper semi-continuous. In fact, since Ω'' is compact, $V_{\Omega'}^{-1}$ is compact valued. If we show that $V_{\Omega'}^{-1}$ is lower semi-continuous, then it follows from Proposition 3 and Proposition 4 that $V_{\Omega'}^{-1}$ is uniformly continuous. To see that $V_{\Omega'}^{-1}$ is lower semi-continuous, we need to show that for any d and x such that $V_{\Omega'}(x) = d$ and any open set U containing x, $V_{\Omega'}(U)$ contains an open neighborhood of d. Since, $V_{\Omega'}$ has non-zero differential, the gradient of $V_{\Omega'}$ in a neighborhood of x is non-zero. Further, the differential of $V_{\Omega'}$ is continuous. Hence, along some direction, the differential of $V_{\Omega'}$ is non-zero and continuous. In that direction, the points in a neighborhood of x map under V to a neighborhood of d.

Note $\bar{0}$ is the unique point mapped to 0 through $V_{\Omega'}$. Also, Ω' contains an open neighborhood all of whose points are in the domain of $V_{\Omega'}$. It can be verified that all the conditions in the semi-completeness of $V_{\Omega'}$ hold.

Next, we show that if V is a Lyapunov function, then it satisfies the conditions in the theorem. Note that since $\dot{V} < 0$

for points other than $\bar{0}$, V has no zero differentials in a small neighborhood around $\bar{0}$. We can choose Ω' as before. The fact that $V_{\Omega'}$ is a uniformly continuous simulation and is semi-complete follows by the same argument as above. Also, the fact that $\mathcal{H}_2$ is Lyapunov stable follows from the above argument. We need to show that every trajectory in $\mathcal{H}_2$ converges to 0. It follows from the fact that every trajectory in $\mathcal{H}_1$ converges to $\bar{0}$ (from Lyapunov's theorem) and V is continuous. □

D. Proof of Theorem 6

Theorem *6:* Given the switched system in Equation (6), let $\bar{V}$ be a multiple weak Lyapunov function for the switched system, such that for each $1 \leq i \leq N$, V_i has a non-zero differential. Then, there exists a neighborhood $\bar{\Omega}$ of $\mathbb{R}_{\geq 0} \times \bar{0}$ such that $\bar{V}[\alpha]_{\bar{\Omega}}$, the restriction of the domain of $\bar{V}[\alpha]$ to $\bar{\Omega}$, satisfies:

- $\bar{V}[\alpha]_{\bar{\Omega}}(\mathcal{H}_{f_1,\ldots,f_N,\alpha})$ is Lyapunov stable.
- $\bar{V}[\alpha]_{\bar{\Omega}}$ is a uniformly continuous simulation which is semi-complete with respect to the sets $\mathcal{T}_{f_1,\ldots,f_N,\alpha,\bar{0}}$ and $\bar{V}[\alpha]_{\bar{\Omega}}(\mathcal{T}_{f_1,\ldots,f_N,\alpha,\bar{0}})$.

Therefore, $\mathcal{H}_{f_1,\ldots,f_N,\alpha}$ is Lyapunov stable.

Proof. For each i, let Ω_i be a compact sub-level set of V_i containing an open neighborhood around the origin such that V_i restricted to Ω_i has non-zero differential. Let $\bar{\Omega}$ be $\cup_i(\mathbb{R}_{\geq 0} \times \Omega_i)$.

First, we show that $\bar{V}[\alpha]_{\bar{\Omega}}(\mathcal{H}_{f_1,\ldots,f_N,\alpha})$ is Lyapunov stable. Let $\epsilon > 0$. Let ϵ' be such that a ball of radius ϵ' is contained in the ϵ-level set for each V_i. Since $\bar{V}$ is a weak Lyapunov function for $\mathcal{H}_{f_1,\ldots,f_N,\alpha}$, it is Lyapunov stable. There exists a δ' such that all executions of $\mathcal{H}_{f_1,\ldots,f_N,\alpha}$ starting in the $\bar{0} \times B_{\delta'}(\bar{0})$ remain within $\{t\} \times B_{\epsilon'}(\bar{0})$ after a total time t has elapsed. Let $\delta_i > 0$ be such that the δ_i-level-set of V_i is contained in $B_{\delta'}(\bar{0})$. Let $\delta = \min_i \delta_i$. Any execution of $\bar{V}[\alpha]_{\bar{\Omega}}(\mathcal{H}_{f_1,\ldots,f_N,\alpha})$ starting in the $\{0\}\times[-\delta, \delta]$ remains within $\{t\} \times [-\epsilon, \epsilon]$ after a total time t has elapsed.

Since every trajectory of $\mathcal{H}_{f_1,\ldots,f_N,\alpha}$ starting in the domain of $\bar{V}[\alpha]_{\bar{\Omega}}$ remains in the domain, its image exists in $\bar{V}[\alpha]_{\bar{\Omega}}(\mathcal{H}_{f_1,\ldots,f_N,\alpha})$, therefore $\bar{V}[\alpha]_{\bar{\Omega}}$ is a simulation. Uniform continuity of the function and its inverse follows from the uniform continuity of the component functions and inverses, namely, V_i and V_i^{-1}. It can be verified that the conditions of semi-completeness are satisfied. □

E. Proof of Theorem 8

Theorem *8:* Let $\dot{x} = F(x)$ be a dynamical system, where F is continuously differentiable, x_0 a hyperbolic equilibrium point, $A = DF(x_0)$ and h a Hartman-Grobman function for $\dot{x} = F(x)$. Let $\mathcal{H}_A$ and $\mathcal{H}_F$ be the hybrid transition systems associated with $\dot{x} = Ax$ and $\dot{x} = F(x)$, respectively. Then, there exists a compact set Ω containing a neighborhood of x_0 such that h_Ω, the restriction of the domain of h to Ω, satisfies:

- h_Ω is a uniformly continuous bisimulation between $\mathcal{H}_A \cap \Omega$ and $\mathcal{H}_F \cap \Omega$ which is semi-complete with respect to $\mathcal{T}_{A,x_0}$ and $\mathcal{T}_{F,x_0}$.

Therefore, $\mathcal{H}_F$ is (Lyapunov) asymptotically stable if and only if $\mathcal{H}_A$ is (Lyapunov) asymptotically stable.

Proof. Take Ω to be any compact subset of the domain of h around the x_0 containing an open neighborhood around x_0. Since h_Ω is a homeomorphism with a compact domain, both h_Ω and h_Ω^{-1} are uniformly continuous functions. Further, it is a bisimulation, since the condition $\varphi(h(x), t) = h(x_0 + e^{At}(x-x_0))$ is equivalent to saying that h maps a trajectory of $\dot{x} = Ax$ starting from x and remaining within Ω, namely, the trajectory mapping time t to $x_0 + e^{At}(x - x_0)$, to a trajectory of $\dot{x} = F(x)$ starting from $h(x)$ and remaining within $h(\Omega)$, namely, $\varphi(h(x), t)$, and vice versa. Finally, it can be verified that the conditions of semi-completeness are satisfied. Note that h maps $\bar{0}$ to $\bar{0}$. □

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