STORMED hybrid games

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Abstract. We introduce STORMED hybrid games (*SHG*), a generalization of STORMED Hybrid Systems [21], which have natural specifications, allow rich continuous dynamics and admit various properties to be decidable. We solve the control problem for *SHG* using a reduction to bisimulation on game graphs. This reduction generalizes to a greater family of games, which includes o-minimal hybrid games [5]. We also solve the optimal-cost reachability problem for Weighted *SHG* and prove decidability of WCTL for Weighted STORMED hybrid systems.

1 Introduction

Hybrid automata are a popular formalism for modelling and verifying embedded systems. *Hybrid games* [13, 5, 6] have been extensively used for modelling and designing *hybrid* controllers to control embedded systems. They are defined similar to hybrid automata but with discrete transitions partitioned into controllable and uncontrollable transitions.

We introduce STORMED hybrid games (*SHG*) defined using STORMED hybrid systems (*SHS*) formalism [21]. *SHG* games allow for richer continuous dynamics than the other popular decidable formalisms like rectangular hybrid games [13] and timed games [2, 3]. Also they admit a stronger coupling between the continuous and discrete state components than found in o-minimal hybrid games [5].

Our main result is to show that for regular winning objectives, the controller synthesis problem is decidable, provided the o-minimal theory used to describe the *SHG* is decidable. The proof depends on the technical observation that under special acyclicity conditions, bisimulation equivalence on the *time-abstract* transition system defined by the *SHG* preserves winning (and losing) states which is not true in general for hybrid systems [5]. The time-abstract transition system is the labelled transition system semantics of the *SHG* that ignores the distinction between controllable and uncontrollable transitions and abstracts the time when continuous transitions are taken. We show that both STORMED systems and o-minimal systems meet this technical acyclicity condition. Further the observations that the time-abstract transition system for a *SHS* has a finite bisimulation quotient [21] (which is effectively constructable when the underlying o-minimal theory is decidable) and the fact that finite games with regular objectives are decidable [18], allow us to conclude the decidability of *SHG*. The same argument holds for o-minimal systems.

Next we consider weighted hybrid games, where there is a cost associated with each of the game choices, and the goal is to design optimal (cost) winning strategies for the controller. We show that weighted *SHS*(WSHS) with reachability objectives are decidable (and the controller synthesizable) when the underlying o-minimal theory is decidable. In order to prove this decidability result, we observe that, when considering non-zeno plays [1], if there is a winning strategy λ for the controller then there is a winning strategy in which the controller does not choose a time step if in the previous step the controller chose a time step shorter than the environment. This technical lemma, while natural, is challenging due to the fact that these games may not have memoryless winning strategies. We then conclude that for non-zeno reachability games for *SHS*, we need only consider bounded step strategies, and therefore can not only compute the cost of the optimal strategy but also synthesize it.

The controller synthesis problem for real-time and hybrid systems has attracted a lot of attention since [2] and [16]. Generally one assumes that a controller can examine the state at various times, and can influence the discrete steps that are taken. Assuming that the controller can observe the state at certain discrete time instants, it has been shown that the controller synthesis problem is decidable for rectangular hybrid automata [14]. In the dense time setting, there are different formulations of the controller synthesis problem. Assuming that the controller can only enable or disable transitions (and not influence when they are taken), it has been shown that the synthesis problem is undecidable for rectangular hybrid automata but decidable for initialized rectangular hybrid automata [13]. When the controller choose not only the transition but also when it is taken, the problem is known to be undecidable for initialized rectangular automata [15], and decidable for timed automata [16], and o-minimal hybrid automata [5] with decidable theories. Here we extend these observations to STORMED systems.

Symbolic algorithms for the controller synthesis problem were first presented in [10]. The controller synthesis problem has also been considered for dynamical systems (those with one discrete state) [11, 20] where dynamical systems is first discretized, and also for switched systems, where the environment has limited power [17]. General categorical conditions on the controller synthesis problem are identified in [12, 19].

Zeno behavior must be dealt in a dense time setting [9]. It is either avoided by imposing syntactic constraints on the game graph [2, 3], restricting the kind of game moves allowed [5, 6], or by semantic constraints imposed on the winning condition [10, 8]. Here we take the latter approach.

Weighted timed games were first considered in [1, 4]. Synthesizing the optimal cost controller for reachability is undecidable for timed automata [7], but decidable for o-minimal hybrid systems [6] with decidable underlying theories. Model checking timed automata against WCTL properties has been shown to be undecidable [7] but decidable for o-minimal systems [6] with decidable underlying theories. We show that optimal for STORMED games and model checking for Weighted STORMED hybrid systems are decidable.

Due to lack of space, we refer the reader to [22] for more details, including proofs and intermediate lemmas and definitions. Hence, we present here only important definitions and results in the following sections.

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[1] In the games we consider, zeno plays are allowed; the environment (or controller) could simply pick shorter and shorter time steps, and thereby starve her opponent.

2 Decidability of control for STORMED Hybrid Games

Definition 1. *A* hybrid game $\mathcal{H}$ *is a tuple* $(\mathit{Loc}, \mathit{Act}_C, \mathit{Act}_U, \mathit{Labels}, \mathit{Cont}, \mathit{Edge}, \mathit{Inv}, \mathit{Flow}, \mathit{Reset}, \mathit{Guard}, \mathit{Lfunc})$ *where:*

- *Loc is a finite set of locations,*
- Act_C *is a finite set of controllable actions,*
- Act_U *is a finite set of uncontrollable actions,*
- *Labels is a finite set of state labels,*
- $\mathit{Cont} = \mathbb{R}^n$ *for some* n*, is a set of continuous states,*
- $\mathit{Edge} \subseteq \mathit{Loc} \times (\mathit{Act}_C \cup \mathit{Act}_U) \times \mathit{Loc}$ *is a set of edges,*
- $\mathit{Inv} : \mathit{Loc} \rightarrow 2^{\mathit{Cont}}$ *is a function that associates with every location an invariant,*
- $\mathit{Flow} : \mathit{Loc} \times \mathit{Cont} \rightarrow (\mathbb{R}^+ \rightarrow \mathit{Cont})$ *is a flow function,*
- $\mathit{Guard} : \mathit{Edge} \rightarrow 2^{\mathit{Cont}}$ *is a function that assigns to each edge a guard,*
- $\mathit{Reset} : \mathit{Edge} \rightarrow 2^{\mathit{Cont} \times \mathit{Cont}}$ *is a reset function, and*
- $\mathit{Lfunc} : \mathit{Loc} \times \mathit{Cont} \rightarrow \mathit{Labels}$ *is a state labeling function.*

At each step of the game, the controller and the environment have two choices: either to let time pass for t time units or to take a controllable (or uncontrollable) transition enabled at the state. If both the controller and the environment pick time, then the system evolves continuously for the shorter of the two durations. If exactly one of them picks a discrete transition, then the discrete transition chosen is taken and finally, in the case when both pick discrete transitions, the controller's choice is respected. A play is an alternating sequence of states and transitions. From each state both the controller and the environment propose a transition, and the transition followed by it in the play is chosen according to the above rule. A strategy for the controller tells the transition that needs to be taken given the information of the play till then. A play conforms to a strategy if the controller selects the transitions according to the strategy. A trace is an alternating sequence of state labels and actions. The trace of a play is the sequence of labels of its states and the actions. A winning condition is a set of admissible traces. A strategy for the controller is winning with respect to a winning condition if the trace of every play conforming with the strategy is admissible according to the winning condition. We consider winning conditions which are ω-regular. The *control problem* is to decide given a hybrid game and a winning condition if the controller has a strategy which is winning. Further, the controller synthesis problem is to come up with such a strategy. The formal semantics of a hybrid game is given in terms of a game graph and can be found in [22].

We now define STORMED hybrid games.

Definition 2. *A* STORMED hybrid game *is defined as a hybrid game with the following restrictions.*

S Guards are *S*eparable:
For all $l_1, l_2 \in \mathit{Loc}$ such that $l_1 \neq l_2$, $\mathit{dist}(\mathcal{G}(l_1), \mathcal{G}(l_2)) = \inf\{||x - y|| \mid x \in \mathcal{G}(l_1), y \in \mathcal{G}(l_2)\} > 0$.

T The flow is time-independent spatially consistent (*T*ISC):
For every state $(l, x) \in \mathit{Loc} \times \mathit{Cont}$, $\mathit{Flow}(l, x)$ is continuous and $\mathit{Flow}(l, x)(0) = x$, and for all $t, t' \in \mathbb{R}^+$, $\mathit{Flow}(l, x)(t + t') = \mathit{Flow}(l, \mathit{Flow}(l, x)(t))(t')$.

O The guards, invariants, flows and resets are definable in an *o*-minimal [2] theory, that is, by a first order formula of the theory.

RM Resets and flows are monotonic along some vector ϕ:
There exists $\epsilon > 0$ such that for all $l \in \mathit{Loc}, x \in \mathit{Cont}$ and $t, \tau \in \mathbb{R}^+$, $\phi \cdot (\mathit{Flow}(l,x)(t+\tau) - \mathit{Flow}(l,x)(t)) \geq \epsilon ||\mathit{Flow}(l,x)(t+\tau) - \mathit{Flow}(l,x)(t)||$.
There exist $\epsilon, \zeta > 0$ such that for all $l_1, l_2 \in \mathit{Loc}$ and $x_1, x_2 \in \mathit{Cont}$ such that $(x_1, x_2) \in \mathit{Reset}(l_1, l_2)$:
- if $l_1 = l_2$, then either $x_1 = x_2$ or $\phi \cdot (x_2 - x_1) \geq \zeta$, and
- otherwise $\phi \cdot (x_2 - x_1) \geq \epsilon ||x_2 - x_1||$.

ED Guards are *e*nds-*d*elimited along ϕ: The set $\{\phi \cdot x \mid x \in \mathcal{G}(l), l \in \mathit{Loc}\} \subseteq [b^-, b^+]$ for some b^- and b^+.

The following theorem states that the control problem is decidable for this class.

Theorem 1. *Given a STORMED hybrid game $\mathcal{H}$ and a winning condition $\mathcal{W}$ which is ω-regular, the control problem is decidable if the underlying o-minimal theory is decidable. The controller synthesis problem is also decidable.*

3 Weighted Hybrid Games and Hybrid Systems

We now consider weighted games, where transitions have associated costs, and the goal is to minimize these costs while meeting certain qualitative objectives. We consider the problem of designing optimal controllers for reachability objectives, and also the problem of verifying hybrid systems with costs.

A *Weighted hybrid game* is a pair $(\mathcal{G}, \mathsf{Cost})$, where the hybrid game $\mathcal{G}$ is equipped with a non-negative and time-non-decreasing cost function $\mathsf{Cost} : \mathit{Loc} \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, i.e., $\mathsf{Cost}(q,t) \geq 0$ for all t and $\mathsf{Cost}(q,t_1) \geq \mathsf{Cost}(q,t_2)$ if $t_1 > t_2$. In addition the cost function satisfies the following additive property $\mathsf{Cost}(q, t_1 + t_2) = \mathsf{Cost}(q,t_1) + \mathsf{Cost}(q,t_2)$. Hence with every move of the controller and the environment, there is an associated cost (the cost associated with a discrete transition can be assumed to be 0). The cost associated with a play is the sum of the cost of all the moves. The cost associated with a controller strategy is the supremum of the costs of all plays conforming with the strategy. Given a Goal, a set of states, a strategy is said to reach the goal if all the plays conforming with the strategy reach the goal. Given a Goal (a set of states), the *optimal cost reachability problem* is to compute the infimum of the costs of the controller strategies which reach goal. If there exists a strategy which achieves the infimum, we call it an optimal strategy.

A *Weighted STORMED hybrid game* is a pair $(\mathcal{G}, \mathsf{Cost})$, where $\mathcal{G}$ is a STORMED hybrid game and the cost function Cost is definable is the o-minimal theory in which $\mathcal{G}$ is defined. The next theorem states that we can solve the optimal-cost reachability problem for Weighted STORMED hybrid games.

Theorem 2. *Given a Weighted STORMED hybrid game* $(\mathcal{G}, \mathsf{Cost})$*, where the underlying theory $\mathcal{M}$ is decidable, and a* Goal*, where* Goal *is definable in $\mathcal{M}$, the optimal-cost*

[2] A theory is o-minimal if the sets definable by formulas with one variable are a finite union of intervals.

reachability problem is decidable. In fact, we can compute an optimal strategy if one exists.

We next turn to model checking Weighted STORMED hybrid systems. There are special instances of Weighted STORMED hybrid games, in which the controller has no choice, i.e., there is exactly one controllable action.

Theorem 3. *Given a Weighted STORMED hybrid system $\mathcal{H}$ definable in a structure $\mathcal{M}$ which is decidable, and a WCTL formula ϕ over $\mathcal{M}$ and Σ_Q where Σ_Q is the set of state labels of game$(\mathcal{H})$, game$(\mathcal{H}), q \models \phi$ is decidable.*

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