On the expressiveness of MTL with past operators

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Abstract. We compare the expressiveness of variants of Metric Temporal Logic (MTL) obtained by adding the past operators 'S' and 'S_I'. We consider these variants under the "pointwise" and "continuous" interpretations over both finite and infinite models. Among other results, we show that for each of these variants the continuous version is strictly more expressive than the pointwise version. We also prove a counter-freeness result for MTL which helps to carry over some results from [3] for the case of infinite models to the case of finite models.

1 Introduction

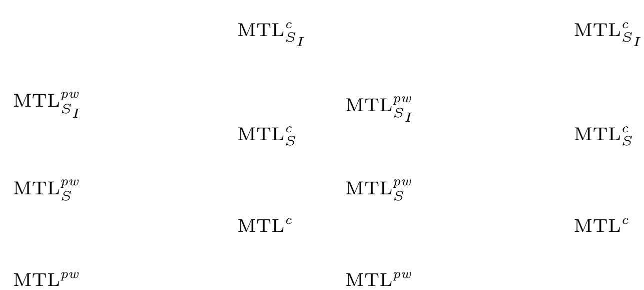
The timed temporal logic *Metric Temporal Logic* (MTL) [6] has received much attention in the literature on the verification of real-time systems. It is interpreted over (finite or infinite) timed behaviours and extends the until operator of classical temporal logic with an interval which specifies the time distance within which the formula must be satisfied. Over dense time the logic has traditionally been interpreted in either of the two ways which have come to be known as the "pointwise" and the "continuous" semantics. In the pointwise version temporal assertions are interpreted only at time points where an "action" or "event" happens in the observed timed behaviour of a system, whereas in the continuous version one is allowed to assert formulas at arbitrary time points between events as well. For instance consider a timed word comprising two events: an a which happens at time 1 and a b which occurs at time 3. Then the MTL formula $\Diamond_{[1,1]}b$ ("a b occurs at a distance of 1 time unit") is not true at any point in this model in the pointwise semantics, since there is no action point from which the action b happens at a distance of 1 time unit. However in the continuous semantics the formula is true at the time instant 2 in the model since at this point the event b occurs at a time distance of 1.

There are many results in the literature regarding the decidability of these logics and the reader is referred to [2, 1, 8, 9] for more details. In this paper we are more interested in the expressiveness of the variants of MTL obtained by adding the past operators S ("since") and S_I (interval constrained "since"), under the pointwise and continuous interpretations, for both finite and infinite models. We will refer to these logics as MTL_S and MTL_{S_I} respectively, and add the superscripts pw and c to denote the pointwise and continuous versions of the logics respectively.

It is easy to see that for each of these variants the continuous version is at least as expressive as the pointwise version, as one can characterize the action points in the continuous semantics, and hence mimic the pointwise interpretation. There have also been some strict containment results. In [3], it is shown that the language L_{2b}, which consists of timed words in which there are two occurrences of b's in the interval $(0, 1)$, is not expressible by MTL in the pointwise semantics but is expressible by MTL in the continuous semantics, and also by MTL_S in the pointwise semantics. It is also shown that the language L_{last_a}, which consists of timed words in which there is an action at time 1 which is preceded by an a, is not expressible by MTL in the continuous semantics but is expressible by MTL_S in the continuous semantics. However these results hold for the case of infinite words and do not extend readily to the case of finite words. The proofs exploit the fact that the models are infinite by using the property that the futures of two distinct points in the constructed models are the same (which is never true for any finite model).

In [4], it is shown that MTL in the continuous semantics is strictly more expressive than MTL in the pointwise semantics for the case of finite words. This is done by showing that the language L_{ni} (for "no insertions") over the alphabet $\{a, b\}$, consisting of timed words in which for every two consecutive a's the time period between them translated by one time unit does not contain any events, is expressible in the continuous semantics, but its expressibility in the pointwise semantics would render the logic undecidable, contradicting the decidability result in [8].

The diagram below shows the known relative expressiveness results. The solid arrows denote "strict containment", the dashed arrows represent "containment", the dashed line says that "relative expressiveness in not known", and the absence of an arrow, transitive arrow, or line, denotes "incomparable".

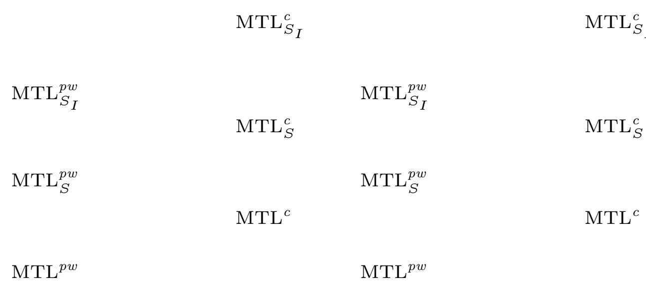


In this paper we first show a way of carrying over the results of [3] to the case of finite words by proving a kind of "counter-freeness" property of MTL. We show that for a given MTL formula φ, there cannot exist finite timed words μ, τ and ν, such that for infinitely many i's, $\mu\tau^i\nu$ is a model of φ, and for infinitely many i's, $\mu\tau^i\nu$ is not a model of φ. This is true for the pointwise semantics and we show a similar result for the continuous semantics which takes into account the "granularity" of φ. These results help us in extending the results of [3] to finite models.

Next we show that each of the continuous versions of the logic is strictly more expressive than its pointwise counterpart. We do so by showing that the language L_{2ins}, which consists of timed words which contain two consecutive a's such that the time period between them when translated by one time unit contains two a's, is not expressible by MTL_{S_I} (and hence by MTL_S and MTL) in the pointwise semantics, but is expressible by MTL (and hence by MTL_S and MTL_{S_I}) in the continuous semantics.

Finally we show that the language L_{em} (for "exact match"), which consists of timed words such that for every a in the interval $(0, 1)$ there is an a in the interval $(1, 2)$ at distance 1 from it, and vice versa, is expressible by MTL_{S_I} in the pointwise semantics but not by MTL_S in the pointwise semantics. This result holds for both finite and infinite words.

The picture below summarizes the relative expressiveness of the various version of MTL after the work in this paper.



We note that it is still open whether $\mathrm{MTL}^c_{S_I}$ is strictly more expressive than MTL^c_S and whether $\mathrm{MTL}^{pw}_{S_I}$ is contained in or incomparable with MTL^c_S.

At some places in the sequel, we have only sketched the proofs due to lack of space. However, the details can be found in [10].

2 Preliminaries

We begin with some preliminary definitions. As usual, A^* and A^ω will denote the set of finite words and the set of infinite words over an alphabet A, respectively. For a finite word $w = a_1 \cdots a_n$ we use $|w|$ to denote the length of w (in this case n). Given finite words u and v, we denote the concatenation of u followed by v as $u \cdot v$, or just uv. We use u^i to denote the concatenation of u with itself i times, and u^ω to denote the infinite word obtained by repeated concatenation of u. We extend these notations to subsets of A^* in the standard way.

The set of non-negative and positive real numbers will be denoted by $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ respectively, the set of positive rational numbers by $\mathbb{Q}_{>0}$, and the set of non-negative integers by $\mathbb{N}$.

We now define finite and infinite timed words which are sequences of action and time pairs. An *infinite timed word* α over an alphabet Σ is an element of $(\Sigma \times \mathbb{R}_{>0})^\omega$ of the form $(a_1, t_1)(a_2, t_2)\cdots$ satisfying:

- (Strict-monotonicity) $t_1 < t_2 < \cdots$.
- (Progressiveness) For every $t \in \mathbb{R}_{\geq 0}$, there exists $i \in \mathbb{N}$ such that $t_i > t$.

Wherever convenient we will also denote the timed word α above as a sequence of delay and action pairs $(d_1, a_1)(d_2, a_2)\cdots$, where for each i, $d_i = t_i - t_{i-1}$. Here and elsewhere we use the convention that t_0 denotes the time point 0.

A *finite timed word* over Σ is an element of $(\Sigma \times \mathbb{R}_{>0})^*$ which satisfies the strict monotonicity condition above. Given $\sigma = (a_1, t_1)(a_2, t_2)\cdots(a_n, t_n)$, we use $time(\sigma)$ to denote the time of the last action, namely t_n. The delay representation for the above finite timed word σ is $(d_1, a_1)\cdots(d_n, a_n)$ where for each i, $d_i = t_i - t_{i-1}$. Given finite timed words σ and ρ, the delay representation for the concatenation of σ followed by ρ is the concatenation of the delay representations of σ and ρ. We will use $T\Sigma^*$ for the set of all finite timed words over Σ, and $T\Sigma^\omega$ for the set of all infinite timed words over Σ.

We now give the syntax and semantics of the two versions of the logic MTL_{S_I}. Let us fix a finite alphabet Σ for the rest of this section. The formulas of MTL_{S_I} over the alphabet Σ are built up from symbols in Σ by boolean connectives and time-constrained versions of the temporal logic operators U ("until") and S ("since"). The formulas of MTL_{S_I} over an alphabet Σ are inductively defined as follows:

$$\varphi := a \mid \neg\varphi \mid (\varphi \vee \varphi) \mid (\varphi U_I \varphi) \mid (\varphi S_I \varphi),$$

where $a \in \Sigma$ and I is an interval with end-points which are rational or ∞.

The models for both the pointwise and continuous interpretations will be timed words over Σ. With the aim of having a common syntax for the pointwise and continuous versions, we use "until" and "since" operators which are "strict" in their first argument.

We first define the *pointwise* semantics for MTL_{S_I} for finite words. Given an MTL_{S_I} formula φ, a finite timed word $\sigma = (a_1, t_1)(a_2, t_2)\cdots(a_n, t_n)$ and a position $i \in \{0, \ldots, n\}$ denoting the leftmost time point 0 or one of the action points $t_1, t_2, \cdots, t_n$, the satisfaction relation $\sigma, i \models_{pw} \varphi$ (read "σ at position i satisfies φ in the pointwise semantics") is inductively defined as:

$$
\begin{array}{lll}
\sigma, i \models_{pw} a & \text{iff} & a_i = a. \\
\sigma, i \models_{pw} \neg\varphi & \text{iff} & \sigma, i \not\models_{pw} \varphi. \\
\sigma, i \models_{pw} \varphi_1 \vee \varphi_2 & \text{iff} & \sigma, i \models_{pw} \varphi_1 \text{ or } \sigma, i \models_{pw} \varphi_2. \\
\sigma, i \models_{pw} \varphi_1 U_I \varphi_2 & \text{iff} & \exists j : i \leq j \leq |\sigma| \text{ such that } t_j - t_i \in I,\ \sigma, j \models_{pw} \varphi_2, \\
 & & \quad \text{and } \forall k \text{ such that } i < k < j,\ \sigma, k \models_{pw} \varphi_1. \\
\sigma, i \models_{pw} \varphi_1 S_I \varphi_2 & \text{iff} & \exists j : 0 \leq j \leq i \text{ such that } t_i - t_j \in I,\ \sigma, j \models_{pw} \varphi_2, \\
 & & \quad \text{and } \forall k \text{ such that } j < k < i,\ \sigma, k \models_{pw} \varphi_1.
\end{array}
$$

The timed language defined by an MTL_{S_I} formula φ in the pointwise semantics over finite timed words is given by $L^{pw}(\varphi) = \{\, \sigma \in T\Sigma^* \mid \sigma, 0 \models_{pw} \varphi \}$. We will use $\mathrm{MTL}_{S_I}^{pw}$ to denote the pointwise interpretation of this logic.

We now turn to the continuous semantics. Given an MTL_{S_I} formula φ, a finite timed word $\sigma = (a_1, t_1)(a_2, t_2)\cdots(a_n, t_n)$ and a time $t \in \mathbb{R}_{\geq 0}$, such that $0 \leq t \leq time(\sigma)$, the satisfaction relation $\sigma, t \models_c \varphi$ (read "σ at time t satisfies φ in the continuous semantics") is inductively defined as follows:

$$
\begin{array}{lll}
\sigma, t \models_c a & \text{iff} & \exists i \text{ such that } t_i = t \text{ and } a_i = a. \\
\sigma, t \models_c \neg\varphi & \text{iff} & \sigma, t \not\models_c \varphi. \\
\sigma, t \models_c \varphi_1 \vee \varphi_2 & \text{iff} & \sigma, t \models_c \varphi_1 \text{ or } \sigma, t \models_c \varphi_2. \\
\sigma, t \models_c \varphi_1 U_I \varphi_2 & \text{iff} & \exists t' \text{ such that } t \leq t' \leq time(\sigma),\ t' - t \in I,\ \sigma, t' \models_c \varphi_2 \\
& & \text{and } \forall t'' \text{ such that } t < t'' < t',\ \sigma, t'' \models_c \varphi_1. \\
\sigma, t \models_c \varphi_1 S_I \varphi_2 & \text{iff} & \exists t' \text{ such that } 0 \leq t' \leq t,\ t - t' \in I,\ \sigma, t' \models_c \varphi_2 \\
& & \text{and } \forall t'' \text{ such that } t' < t'' < t,\ \sigma, t'' \models_c \varphi_1.
\end{array}
$$

The timed language defined by an MTL_{S_I} formula φ in the continuous semantics over finite timed words is defined as $L^c(\varphi) = \{\, \sigma \in T\Sigma^* \mid \sigma, 0 \models_c \varphi \}$. We will use $\mathrm{MTL}^c_{S_I}$ to denote this continuous interpretation of the MTL_{S_I} formulas.

We can similarly define the semantics for infinite timed words. The only change would be to replace $time(\sigma)$ and $|\sigma|$ by ∞.

We define the following derived operators which we will make use of in the sequel. Syntactically, $\Diamond_I\varphi$ is $\top U_I \varphi$, $\Box_I\varphi$ is $\neg\Diamond_I\neg\varphi$, $\varphi_1 U \varphi_2$ is $\varphi_1 U_{(0,\infty)} \varphi_2$, $\Diamond\varphi$ is $\Diamond_{(0,\infty)}\varphi$, $\Box\varphi$ is $\neg\Diamond\neg\varphi$, $\diamondminus_I\varphi$ is $\top S_I \varphi$, $\boxminus_I\varphi$ is $\neg\diamondminus_I\neg\varphi$, $\varphi_1 S \varphi_2$ is $\varphi_1 S_{(0,\infty)} \varphi_2$, $\diamondminus\varphi$ is $\diamondminus_{(0,\infty)}\varphi$ and $\boxminus\varphi$ is $\neg\diamondminus\neg\varphi$.

The fragment of MTL_{S_I} without the S_I operator will be called MTL. The fragment of MTL_{S_I} obtained by replacing S_I with the derived operator S will be called MTL_S. We denote their pointwise and continuous interpretations by MTL^{pw} and MTL^c, and MTL^{pw}_S and MTL^c_S respectively. The continuous versions of the above logics can be seen to be at least as expressive as their pointwise versions. This is because one can characterize the occurrence of an action point in a timed word in the continuous semantics using the formula $\varphi_{act} = \bigvee_{a \in \Sigma} a$. We can then force assertions to be interpreted only at these action points.

3 Ultimate satisfiability of MTL^{pw}

In this section we show that an MTL formula in the pointwise semantics is either ultimately satisfied or ultimately not satisfied over a periodic sequence of timed words, leading to a "counter-freeness" property of MTL.

We first define the notion of when a formula is *ultimately satisfied* or *ultimately not satisfied* over a sequence of finite timed words. Let $\langle\sigma_i\rangle$ be a sequence of finite timed words $\sigma_0, \sigma_1, \cdots$. Given a $j \in \mathbb{N}$ and $\varphi \in \mathrm{MTL}$, we say that $\langle\sigma_i\rangle$ at j *ultimately satisfies* φ, denoted $\langle\sigma_i\rangle, j \models_{us} \varphi$, iff $\exists k \in \mathbb{N} : \forall k' \geq k, \sigma_{k'}, j \models_{pw} \varphi$. We say that $\langle\sigma_i\rangle$ at j *ultimately does not satisfy* φ, denoted $\langle\sigma_i\rangle, j \models_{un} \varphi$, iff $\exists k \in \mathbb{N} : \forall k' \geq k, \sigma_{k'}, j \models_{pw} \neg\varphi$. We refer to the least such k in either case above as the *stability point* of φ at j in $\langle\sigma_i\rangle$.

We now define a *periodic sequence* of timed words. A sequence $\langle\sigma_i\rangle$ of finite timed words is said to be periodic if there exist finite timed words μ, τ and ν, where $|\tau| > 0$, such that $\sigma_i = \mu\tau^i\nu$ for all $i \in \mathbb{N}$.

The following theorem says that a periodic sequence of timed words at a position j either ultimately satisfies a given MTL formula or ultimately does not satisfy it. This is not true in general for a non-periodic sequence. For example, consider the sequence $\langle\sigma_i\rangle$ given by $\sigma_0 = (1, a)$, $\sigma_1 = (1, a)(1, b)$, $\sigma_2 = (1, a)(1, b)(1, a)$, etc. Then the formula $\Diamond(a \wedge \neg\Diamond\top)$, which says that the last action of the timed word is an a, is neither ultimately satisfied nor ultimately not satisfied in $\langle\sigma_i\rangle$ at 0.

Theorem 1. *Let $\langle\sigma_i\rangle$ be a periodic sequence of finite timed words. Let φ be an* MTL *formula and let $j \in \mathbb{N}$. Then either $\langle\sigma_i\rangle, j \models_{us} \varphi$ or $\langle\sigma_i\rangle, j \models_{un} \varphi$.*

Proof. Since $\langle\sigma_i\rangle$ is periodic, there exist timed words $\mu = (d_1, a_1)\cdots(d_l, a_l)$, $\tau = (e_1, b_1)\cdots(e_m, b_m)$ and $\nu = (f_1, c_1)\cdots(f_n, c_n)$, such that $\sigma_i = \mu\tau^i\nu$. Let $\mu\tau^\omega = (a'_1, t'_1)(a'_2, t'_2)\cdots$. We use induction on the structure of φ.

Case $\varphi = a$: If $a'_j = a$, then clearly $\langle\sigma_i\rangle, j \models_{us} \varphi$, otherwise $\langle\sigma_i\rangle, j \models_{un} \varphi$.

Case $\varphi = \neg\psi$: If $\langle\sigma_i\rangle, j \models_{us} \psi$, then $\langle\sigma_i\rangle, j \models_{un} \varphi$. Otherwise, by induction hypothesis, $\langle\sigma_i\rangle, j \models_{un} \psi$ and hence $\langle\sigma_i\rangle, j \models_{us} \varphi$.

Case $\varphi = \eta\vee\psi$: Suppose $\langle\sigma_i\rangle, j \models_{us} \eta$ or $\langle\sigma_i\rangle, j \models_{us} \psi$. Let k be the maximum of the stability points of η and ψ at j in $\langle\sigma_i\rangle$. For all $k' \geq k, \sigma_{k'}, j \models_{pw} \eta$ or for all $k' \geq k, \sigma_{k'}, j \models_{pw} \psi$, and hence for all $k' \geq k, \sigma_{k'}, j \models_{pw} \eta \vee \psi$. Therefore, $\langle\sigma_i\rangle, j \models_{us} \eta\vee\psi$. Otherwise, it is not the case that $\langle\sigma_i\rangle, j \models_{us} \eta$ and it is not the case that $\langle\sigma_i\rangle, j \models_{us} \psi$. By induction hypothesis, $\langle\sigma_i\rangle, j \models_{un} \eta$ and $\langle\sigma_i\rangle, j \models_{un} \psi$. So, $\langle\sigma_i\rangle, j \models_{us} \neg\eta$ and $\langle\sigma_i\rangle, j \models_{us} \neg\psi$. Let k be the maximum of the stability points of $\neg\eta$ and $\neg\psi$ above, at j. Then for all $k' \geq k, \sigma_{k'}, j \models_{pw} \neg\eta$ and for all $k' \geq k, \sigma_{k'}, j \models_{pw} \neg\psi$, and hence for all $k' \geq k, \sigma_{k'}, j \models_{pw} \neg\eta \wedge \neg\psi$. Therefore, $\langle\sigma_i\rangle, j \models_{us} \neg\eta \wedge \neg\psi$ and hence $\langle\sigma_i\rangle, j \models_{us} \neg(\eta \vee \psi)$. So, $\langle\sigma_i\rangle, j \models_{un} \eta \vee \psi$.

Case $\varphi = \eta U_I \psi$: We consider two cases, one in which there exists $j' \geq j$ such that $t_{j'} - t_j \in I$ and $\langle\sigma_i\rangle, j' \models_{us} \psi$ and the other in which the above condition does not hold.

Suppose there exists $j' \geq j$ such that $t_{j'} - t_j \in I$ and $\langle\sigma_i\rangle, j' \models_{us} \psi$. Let j_s be the smallest such j'.

Now suppose for all k such that $j < k < j_s$, $\langle\sigma_i\rangle, k \models_{us} \eta$. Let n_k be the stability point of η at k for each k above and n_{j_s} that of ψ at j_s. Let n' be the maximum of all n_k's and n_{j_s}. So, for all $n'' \geq n'$, $\sigma_{n''}, j \models_{pw} \eta U_I \psi$. Hence $\langle\sigma_i\rangle, j \models_{us} \varphi$.

Otherwise there exists k such that $j < k < j_s$ and $\langle\sigma_i\rangle, k \models_{un} \eta$. Let m_k be the stability point of η at k. For each $j < k' < j_s$ such that $t_{k'} - t_j \in I$, $\langle\sigma_i\rangle, k' \models_{un} \psi$ (because we chose j_s to be the smallest). Let $n_{k'}$ be the stability point of each k' above. Take n' to be the maximum of m_k and $n_{k'}$'s. For all $n'' \geq n'$, $\sigma_{n''}, j \not\models_{pw} \eta U_I \psi$. Hence $\langle\sigma_i\rangle, j \models_{un} \varphi$.

Now turning to the second case, suppose that for all $j' \geq j$ such that $t_{j'} - t_j \in I$, it is not the case that $\langle\sigma_i\rangle, j' \models_{us} \psi$. Then by induction hypothesis, $\langle\sigma_i\rangle, j' \models_{un} \psi$.

Suppose I is bounded. If there is no j' such that $t_{j'} - t_j \in I$, then it is easy to see that $\langle\sigma_i\rangle, j \models_{un} \eta U_I \psi$. Otherwise there exist finite number of j''s which satisfy $t_{j'} - t_j \in I$ and $\langle\sigma_i\rangle, j' \models_{un} \psi$ since I is bounded. Let $n_{j'}$ be the stability point of ψ at each of these $n_{j'}$'s. Take n' to be the maximum of all $n_{j'}$'s. Then for all $n'' \geq n'$, $\sigma_{n''}, j \not\models_{pw} \eta U_I \psi$. Hence $\langle \sigma_i \rangle, j \models_{un} \varphi$.

Suppose I is unbounded. Let $S = \{s_1, s_2, \cdots, s_m\}$ be the suffixes of τ in the order of decreasing length. Thus $s_i = (e_i, b_i) \cdots (e_m, b_m)$. Let $W = \{w_1, w_2, \cdots, w_n\}$ be the suffixes of ν in the order of decreasing length. Let $X = W \cup (S \cdot \tau^* \cdot \nu)$. (We note that we can arrange the timed words in X in the increasing order of length such that the difference in lengths of the adjacent words in this sequence is one and that the succeeding string in the sequence is a prefix of the present. The sequence is $w_n, w_{n-1}, \cdots, w_1, s_n\nu, s_{n-1}\nu, \cdots, s_1\nu, s_n\tau\nu, \cdots, s_1\tau\nu, s_n\tau^2\nu, \cdots, s_1\tau^2\nu$, and so on.)

We now claim that ψ is satisfied at 1 for only finitely many timed words from X. Otherwise ψ is satisfied at 1 for infinitely many timed words from $W \cup S \cdot \tau^* \cdot \nu$ and hence for infinitely many from $s_i \cdot \tau^* \cdot \nu$ for some i. By induction hypothesis $\langle \sigma_i \rangle, l + i \models_{us} \psi$ (l is the length of μ) and therefore $\langle \sigma_i \rangle, l + i + cm \models_{us} \psi$, m is the length of τ and $c \in \mathbb{N}$. Since I is unbounded there exists $j' \geq j$ such that $t_{j'} - t_j \in I$ and $\langle \sigma_i \rangle, j' \models_{us} \psi$. This is a contradiction.

Every $j'' > j$ such that $t_{j''} - t_i \in I$ and $j'' < |\mu|$, $\langle \sigma_i \rangle, j'' \models_{un} \psi$ (by the assumption of the present case). Let $n_{j''}$ be the stability point of the j'''s (which are finite in number).

Suppose there is no timed word in X which satisfies ψ at 1. Let n' be the maximum of $n_{j''}$'s. For all $n'' \geq n'$, $\sigma_{n''}, j \not\models_{pw} \eta U_I \psi$. Hence $\langle \sigma_i \rangle, j \models_{un} \eta U_I \psi$.

Suppose there exists a timed word in X which satisfies ψ at 1. Since we proved that they are finite in number, let l' be the length of the largest such timed word.

Suppose that there exists a timed word in X whose length is greater than l' and which does not satisfy η at 1. Let the length of one such timed word be l''. Let n' be a number which is greater than or equal to the maximum of $n_{j''}$'s and which satisfies $|\sigma_{n'}| > max(j, |\mu|) + l''$. Now for all $n'' \geq n'$, $\sigma_{n''}, j \not\models_{pw} \eta U_I \psi$ since the smallest $j' \geq j$ where ψ is satisfied is $|\sigma_{n''}| - l'$ but before that there is the point $|\sigma_{n''}| - l''$ where η is not satisfied. Hence $\langle \sigma_i \rangle, j \models_{un} \varphi$.

Suppose that all timed words in X whose length is greater than l' satisfy η at 1. Now if there exists $j < k \leq |\mu|$ such that $\langle \sigma_i \rangle, k \models_{un} \eta$, then let n' be such that it is larger than the $n_{j''}$'s and the stability point of η at k and $|\sigma_{n'}| > |\mu| + l'$. For all $n'' \geq n'$, $\sigma_{n''}, j \not\models_{pw} \eta U_I \psi$. Hence, $\langle \sigma_i \rangle, j \models_{un} \varphi$. Otherwise for every $j < k \leq |\mu|$, $\langle \sigma_i \rangle, k \models_{us} \eta$. Take n' to be greater than the maximum of the stability point of η at k's and such that $|\sigma_{n'}| > j + n_I + l'$, where n_I is such that $t_{j+n_I} - t_j \in I$. For all $n'' \geq n'$, $\sigma_{n''}, j \models_{pw} \eta U_I \psi$ and hence $\langle \sigma_i \rangle, j \models_{us} \varphi$. □

It is well known that linear-time temporal logic (LTL) and counter-free languages [5, 7] are expressively equivalent. We recall that a *counter* in a deterministic finite automaton is a finite sequence of states $q_0 q_1 \cdots q_n$ such that $n > 1$, $q_0 = q_n$ and there exists a non-empty finite word v such that every q_i on reading v reaches q_{i+1} for $i = 0, \cdots, n-1$. A counter-free language is a regular language whose minimal DFA does not contain any counters. It is not difficult to see that the following is an equivalent characterization of counter-free languages. A regular language L is a counter-free language iff there do not exist finite words u, v

and w, where $|v| > 0$, such that $uv^iw \in L$ for infinitely many i's and $uv^iw \notin L$ for infinitely many i's.

We show a similar necessary property for timed languages defined by MTL^{pw} formulas. Let us call a timed language L *counter-free* if there do not exist finite timed words μ, τ and ν, where $|\tau| > 0$, such that $\mu\tau^i\nu \in L$ for infinitely many i's and $\mu\tau^i\nu \notin L$ for infinitely many i's. The following theorem follows from the ultimate satisfiability result for MTL^{pw}.

Theorem 2. *Every timed language of finite words definable in* MTL^{pw} *is counter-free.*

Proof. Suppose that a timed language L is definable in MTL^{pw} by a formula φ, but is not counter-free. Then there exist finite timed words μ, τ and ν, where $|\tau| > 0$, such that $\mu\tau^i\nu \in L$ for infinitely many i's and $\mu\tau^i\nu \notin L$ for infinitely many i's. The periodic sequence $\langle\sigma_i\rangle$ where $\sigma_i = \mu\tau^i\nu$, is neither ultimately satisfied by φ nor ultimately not satisfied by it which is a contradiction to Theorem 1. $\square$

The above theorem, for example, implies that the language L_{even_b}, which consists of timed words in which the number of b's is even, is not expressible in MTL^{pw}. Taking $\mu = \nu = \epsilon$ and $\tau = (1, b)$ implies that L_{even_b} is not counter-free. By Theorem 2, L_{even_b} is not definable in MTL^{pw}.

4 Ultimate Satisfiability of MTL^c

In this section we show an ultimate satisfiability result for the continuous semantics analogous to the one in the previous section for pointwise semantics. We show that an MTL^c formula with "granularity" p is either ultimately satisfied or ultimately not satisfied by a "p-periodic" sequence of finite timed words.

We say that an MTL formula φ has *granularity* p, where $p \in \mathbb{Q}_{>0}$, if all the end-points of the intervals in it are either integral multiples of p, or ∞. We denote by $\mathrm{MTL}(p)$ the set of MTL formulas with granularity p. A periodic sequence of timed words $\langle\sigma_i\rangle$ has *period* p if there exist μ, τ and ν such that $time(\tau) = p$ and for each i, $\sigma_i = \mu\tau^i\nu$. Note that every periodic sequence has a unique period.

We now proceed to define the notion of ultimate satisfiability for the continuous semantics. Given a sequence $\langle\sigma_i\rangle$ of finite timed words, $t \in \mathbb{R}_{\geq 0}$ and $\varphi \in \mathrm{MTL}$, we say that $\langle\sigma_i\rangle$ at t *ultimately satisfies* φ in the continuous semantics, denoted $\langle\sigma_i\rangle, t \models^c_{us} \varphi$, iff $\exists j : \forall k \geq j, \sigma_k, t \models_c \varphi$. And we say that $\langle\sigma_i\rangle$ at t *ultimately does not satisfy* φ in the continuous semantics, denoted $\langle\sigma_i\rangle, t \models^c_{un} \varphi$, iff $\exists j : \forall k \geq j, \sigma_k, t \models_c \neg\varphi$.

In the proof of the ultimate satisfiability for the pointwise case in the previous section, we extensively use the argument that if a formula is ultimately satisfied at all points in a bounded interval then there is a point in the periodic sequence after which all timed words in the sequence satisfy the formula at all points in the interval. However the argument fails in the continuous semantics since there are infinitely many time points even in a bounded interval. Towards tackling this problem, we define a *canonical* set of time points in a timed word such that the satisfiability of a formula is invariant between two consecutive points in the set. So given a finite timed word $\sigma = (a_1, t_1) \cdots (a_n, t_n)$ and a $p \in \mathbb{Q}_{>0}$, we define the set of *canonical points* in σ with respect to p to be the set containing 0 and $\{t \mid \exists i, c \in \mathbb{N} : t = t_i - cp\}$. Since this is finite, we can arrange the time points in it in increasing order to get the sequence $r_0 r_1 \cdots r_m$ which we call the *canonical sequence* of σ with respect to p. We mention below some of the immediate properties of a canonical sequence which we will use later.

1. For each $i \in \{0, \cdots, m-1\}$ σ does not contain any action in the interval (r_i, r_{i+1}).
2. Let $t, t' \in [0, \mathit{time}(\sigma)]$ with $t < t'$, such that (t, t') does not contain any r_i. Then for every $c \in \mathbb{N}$, the interval $(t + cp, t' + cp)$ also does not contain any r_i.

Lemma 1. *Let σ be a finite timed word and $p \in \mathbb{Q}_{>0}$. Let $r_0 r_1 \cdots r_m$ be the canonical sequence of σ with respect to p. Let $\varphi \in \mathrm{MTL}(p)$. Then for each $i \in \{0, \cdots, m-1\}$ and for all $t, t' \in (r_i, r_{i+1})$, $\sigma, t \models_c \varphi$ iff $\sigma, t' \models_c \varphi$.* □

With each finite word σ we associate a sequence of delays which specifies the delays between the consecutive canonical points in the canonical sequence. So given a canonical sequence $r_0 r_1 \cdots r_m$ of σ with respect to p, we call the sequence of delays $D = e_1 e_2 \cdots e_m$ an *invariant delay sequence* of σ with respect to p if each $e_i = r_i - r_{i-1}$. Given any subword of σ, $(d_i, a_i) \cdots (d_j, a_j)$, we can associate a delay sequence with it in a natural way which is given by $e_{i'} \cdots e_{j'}$, where i' and j' are such that $\sum_{k=1}^{i-1} d_k = \sum_{k=1}^{i'-1} e_k$ and $\sum_{k=i}^{j} d_k = \sum_{k=i'}^{j'} e_k$.

Proposition 1. *Let $\sigma = \mu\tau\nu$ be a finite timed word such that $\mathit{time}(\tau) = p$ and $p \in \mathbb{Q}_{>0}$. Let $D = D_1 D_2 D_3$ be the invariant delay sequence of σ with respect to p where D_1, D_2 and D_3 are the delay sequences corresponding to the subwords μ, τ and ν. Then for any j, the invariant delay sequence of $\mu\tau^j\nu$ with respect to p is $D_1 (D_2)^j D_3$.* □

To mimic the proof of the pointwise case we define intervals in a timed word in which the satisfaction of formulas is invariant. Moreover we require this breaking up of the timed word into intervals to be consistent in some sense over the timed words in a periodic sequence. Hence we introduce the following definitions.

Given a canonical sequence $r_0 r_1 \cdots r_m$ of σ with respect to p, we define the *invariant interval sequence* of σ with respect to p to be $J = J_0 J_1 \cdots J_{2m}$ where $J_{2i} = [r_i, r_i]$ and $J_{2i+1} = (r_i, r_{i+1})$. It follows from Lemma 1 that the satisfiability of an $\mathrm{MTL}(p)$ formula is invariant in σ over each interval J_i.

Given a delay sequence $D = d_1 \cdots d_m$, we can associate an interval sequence $J = J_0 J_1 \cdots J_{2m}$ with it such that $J_0 = [0, 0]$, $J_{2i} = [t, t]$, where $t = \sum_{j=1}^{i} d_j$ and $J_{2i+1} = (t_1, t_2)$, where $t_1 = \sum_{j=1}^{i} d_j$ and $t_2 = \sum_{j=1}^{i+1} d_j$. Note that the interval sequence associated with an invariant delay sequence is the invariant interval sequence.

Lemma 2. *Let $\sigma = \mu\tau\nu$ be a finite timed word such that $time(\tau) = p$, where $p \in \mathbb{Q}_{>0}$. Let $D = D_1 D_2 D_3$ be the invariant delay sequence of σ with respect to p, where D_1, D_2 and D_3 are the delay sequences corresponding to the subwords μ, τ and ν. Let $\langle\sigma_i\rangle$ be the periodic sequence of finite timed words given by $\sigma_i = \mu\tau^i\nu$. Let $J = J_0 J_1 \cdots$ be the interval sequence corresponding to the delay sequence $D_1(D_2)^\omega$. Then for all $t \in J_j$ and $\varphi \in \mathrm{MTL}(p)$,*

1. *if $\langle\sigma_i\rangle, t \models^c_{us} \varphi$ then there exists n_j such that for all $n \geq n_j$ and $t' \in J_j$, $\sigma_n, t' \models_c \varphi$ and*
2. *if $\langle\sigma_i\rangle, t \models^c_{un} \varphi$ then there exists n_j such that for all $n \geq n_j$ and $t' \in J_j$, $\sigma_n, t' \not\models_c \varphi$.*

□

Proposition 2. *Let $\langle\sigma_i\rangle$ be the periodic sequence given by $\sigma_i = \mu\tau^i\nu$ and $time(\tau) = p$, where $p \in \mathbb{Q}_{>0}$. Let $t \in \mathbb{R}_{\geq 0}$ such that $t > time(\mu)$. Let $c \in \mathbb{N}$ and let $\varphi \in \mathrm{MTL}$. Then $\langle\sigma_i\rangle, t \models^c_{us} \varphi$ iff $\langle\sigma_i\rangle, t + cp \models^c_{us} \varphi$.* □

Theorem 3. *Let $\langle\sigma_i\rangle$ be a periodic sequence with period p, where $p \in \mathbb{Q}_{>0}$. Let φ be an $\mathrm{MTL}(p)$ formula and let $t \in \mathbb{R}_{\geq 0}$. Then either $\langle\sigma_i\rangle, t \models^c_{us} \varphi$ or $\langle\sigma_i\rangle, t \models^c_{un} \varphi$.*

Proof. The proof follows that for the pointwise case. Since the ultimate satisfiability of a formula is invariant within the intervals of an invariant interval sequence, and there exists an n_j for each interval J_j as given by Lemma 2, we consider each of these intervals as one entity (comparable to a point in the pointwise case). The details of the proof can be found in [10]. □

The above theorem gives us a counter-freeness result for the continuous case. Given a $p \in \mathbb{R}_{>0}$, we call a timed language L, *p-counter-free*, if there do not exist timed words μ, τ and ν such that $time(\tau) = p$ and there exist infinitely many i's for which $\mu\tau^i\nu \in L$ and infinitely many of them for which $\mu\tau^i\nu \notin L$. Below is the result for the continuous semantics.

Theorem 4. *Let $p \in \mathbb{Q}_{>0}$. Then every timed language of finite words definable by an $\mathrm{MTL}(p)$ formula in the continuous semantics is p-counter-free.* □

5 Strict containment of MTL^{pw} in MTL^c

In this section we show the strict containment of MTL^{pw} in MTL^c for finite words. We show that the language L_{2b} described below is not expressible by any MTL^{pw} formula. We will first sketch a proof of the same for infinite words. It is a simplified version of the proof in [3] and the details can be found in [10].

L_{2b} is the timed language over the alphabet $\Sigma = \{b\}$ which consists of timed words in which there are at least two b's in the interval $(0, 1)$. Formally, $L_{2b} = \{(b, t_1)(b, t_2) \cdots \in T\Sigma^\omega \mid \exists t_i, t_j : 0 < t_i < t_j < 1\}$.

Let $p \in \mathbb{Q}_{>0}$, where $p = 1/k$ and $k \in \mathbb{N}$. We give two models, α^p and β^p such that $\alpha^p \in L_{2b}$ but $\beta^p \notin L_{2b}$, and no $\mathrm{MTL}(p)$ formula φ can distinguish between the two models in the pointwise semantics in the sense that $\alpha^p, 0 \models_{pw} \varphi$ iff $\beta^p, 0 \models_{pw} \varphi$. α^p is given by the timed word $(1-3p/4, b)(p/2, b)(p/2, b)(p/2, b)\cdots$ and β^p is given by $(1-p/4, b)(p/2, b)(p/2, b)\cdots$. They are depicted below.

α^p

$1-p$ $\quad$ 1 $\quad$ $1+p$ $\quad$ $1+2p$ $\quad$ $1+3p$

β^p

$1-p$ $\quad$ 1 $\quad$ $1+p$ $\quad$ $1+2p$ $\quad$ $1+3p$

Proposition 3. *Let $i, j \in \mathbb{N}$ and $i, j > 0$. Let $\varphi \in$ MTL. Then $\alpha^p, i \models_{pw} \varphi$ iff $\alpha^p, j \models_{pw} \varphi$. Similarly, $\beta^p, i \models_{pw} \varphi$ iff $\beta^p, j \models_{pw} \varphi$ and $\alpha^p, i \models_{pw} \varphi$ iff $\beta^p, j \models_{pw} \varphi$.*

Proof. All proper suffixes of the two timed words are the same. □

Theorem 5. *For any $\varphi \in \mathrm{MTL}(p)$, $\alpha^p, 0 \models_{pw} \varphi$ iff $\beta^p, 0 \models_{pw} \varphi$.* □

Now suppose that there exists an MTL formula φ which in the pointwise semantics defines the language L_{2b}. It belongs to $\mathrm{MTL}(p)$ where $p = 1/k$ and k is the least common multiple of the denominators of the interval end-points in φ (recall that the end points are rational). But φ cannot distinguish between α^p and β^p. It is either satisfied by both of them or is not satisfied by any of them. In either case it does not define L_{2b}. Hence L_{2b} is not definable in MTL^{pw}.

But the disjunction of the formulas, $\Diamond_{(0,0.5]} b \wedge \Diamond_{(0.5,1)} b$, $\Diamond_{(0,0.5]}(b \wedge \Diamond_{(0,0.5)} b)$ and $\Diamond_{(0,0.5)}(\Diamond_{[0.5,0.5]} b \wedge \Diamond_{(0,0.5)} b)$, expresses L_{2b} in the continuous semantics. So, MTL^{pw} is strictly contained in MTL^c over infinite words.

Now we extend the above proof for the case of finite words using the notion of ultimate satisfiability. We replace every infinite word by an infinite sequence of finite timed words. Thus we replace α^p by $\langle \sigma_i^p \rangle$ and β^p by $\langle \rho_i^p \rangle$, respectively, which are defined as $\sigma_i^p = \mu_1 \tau^i$ and $\rho_i^p = \mu_2 \tau^i$, where $\mu_1 = (1-3p/4, b)(1-p/4, b)$, $\mu_2 = (1-p/4, b)$ and $\tau = (p/2, b)$. It can be seen that $\langle \sigma_i^p \rangle$ is completely contained in L_{2b} and $\langle \rho_i^p \rangle$ is completely outside L_{2b}. We can now argue that a formula φ in $\mathrm{MTL}(p)$ is ultimately satisfied at 0 in $\langle \sigma_i^p \rangle$ iff it is ultimately satisfied at 0 in $\langle \rho_i^p \rangle$. We see that the claims which were true for the infinite case continue to hold for the finite case with the notion of ultimate satisfiability.

Proposition 4. *Let $i, j \in \mathbb{N}$ and $i, j > 0$. Let $\varphi \in$ MTL. Then $\langle \sigma_i^p \rangle, i \models_{us} \varphi$ iff $\langle \sigma_i^p \rangle, j \models_{us} \varphi$. Similarly, $\langle \rho_i^p \rangle, i \models_{us} \varphi$ iff $\langle \rho_i^p \rangle, j \models_{us} \varphi$ and $\langle \sigma_i^p \rangle, i \models_{us} \varphi$ iff $\langle \rho_i^p \rangle, j \models_{us} \varphi$.* □

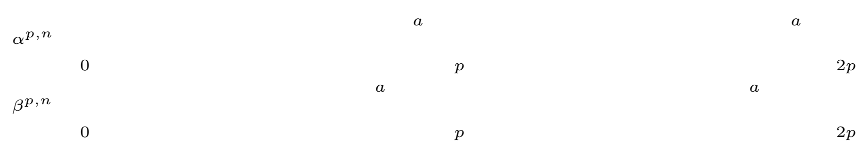
Theorem 6. *Given any $\varphi \in \mathrm{MTL}(p)$, $\langle \sigma_i^p \rangle, 0 \models_{us} \varphi$ iff $\langle \rho_i^p \rangle, 0 \models_{us} \varphi$.* □

Suppose there exists a formula φ which defines L_{2b} in the pointwise semantics. Then $\varphi \in \mathrm{MTL}(p)$ for some p. Since φ defines L_{2b} it is satisfied by all timed words in $\langle \sigma_i^p \rangle$. So φ is ultimately satisfied at 0 in $\langle \sigma_i^p \rangle$ and hence is ultimately satisfied at 0 in $\langle \rho_i^p \rangle$. This is a contradiction since none of the timed words in $\langle \rho_i^p \rangle$ are in L_{2b}. Therefore no MTL formula defines L_{2b} in the pointwise semantics.

6 Strict containment of MTL^c in MTL^c_S

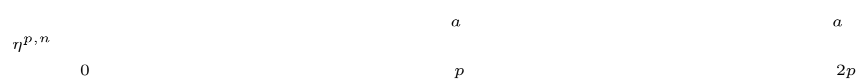
In this section we show that MTL^c is strictly contained in MTL^c_S for finite timed words by showing that the language L_{last_a} is not expressible by any MTL^c formula but is expressible by an MTL^c_S formula. L_{last_a} consists of timed words over $\{a, b\}$ such that the last symbol in the interval $(0, 1)$ is an a and there is an action at time 1. We will sketch a proof of the above claim for the case of infinite words which essentially follows the one given in [3] and then show how it can be extended for finite words.

Let $p \in \mathbb{Q}_{>0}$, where $p = 1/q$ and $q \in \mathbb{N}$, and let $n \in \mathbb{N}$. We give two infinite timed words $\alpha^{p,n}$ and $\beta^{p,n}$ such that $\alpha^{p,n} \in L_{last\_a}$ and $\beta^{p,n} \notin L_{last\_a}$. Let $d = p/(n+4)$. Then,



$\alpha^{p,n} = (c_1, d)(c_2, 2d) \cdots$ where $c_k = a$ if $k \mod (n+4) = n+3$, $c_k = b$ otherwise.
$\beta^{p,n} = (c_1, d)(c_2, 2d) \cdots$ where $c_k = a$ if $k \mod (n+4) = n+2$, $c_k = b$ otherwise.

Let us consider the following infinite model $\eta^{p,n}$ given by the timed word $(c_1, d)(c_2, 2d) \cdots$, where $c_k = a$ if $k \mod (n+4) = 0$, $c_k = b$ otherwise.



We denote by $\text{MTL}(p, k)$ the formulas in $\text{MTL}(p)$ with an U nesting depth of k.

Lemma 3. *Let $k \in \mathbb{N}$ and $0 \leq k \leq n$. Let $\varphi \in \text{MTL}(p, k)$. Let $i, j \in \{1, \cdots, n+3-k\}$ and let $\alpha \geq 0$. Then $\eta^{p,n}, (\alpha(n+4)+i)d \models_c \varphi$ iff $\eta^{p,n}, (\alpha(n+4)+j)d \models_c \varphi$ and for all $t_1, t_2 \in (0, d)$, $\eta^{p,n}, (\alpha(n+4)+i)d - t_1 \models_c \varphi$ iff $\eta^{p,n}, (\alpha(n+4)+j)d - t_2 \models_c \varphi$.* □

Corollary 1. *Let $\varphi \in \text{MTL}(p, n)$ and let $\alpha \geq 0$. Then $\eta^{p,n}, (\alpha(n+4)+1)d \models_c \varphi$ iff $\eta^{p,n}, (\alpha(n+4)+2)d \models_c \varphi$ iff $\eta^{p,n}, (\alpha(n+4)+3)d \models_c \varphi$ and for all $t_1, t_2, t_3 \in (0, d)$, $\eta^{p,n}, (\alpha(n+4)+1)d - t_1 \models_c \varphi$ iff $\eta^{p,n}, (\alpha(n+4)+2)d - t_2 \models_c \varphi$ iff $\eta^{p,n}, (\alpha(n+4)+3)d - t_3 \models_c \varphi$.* □

Theorem 7. *Let $\varphi \in \text{MTL}(p, n)$. Then $\alpha^{p,n}, 0 \models_c \varphi$ iff $\beta^{p,n}, 0 \models_c \varphi$.* □

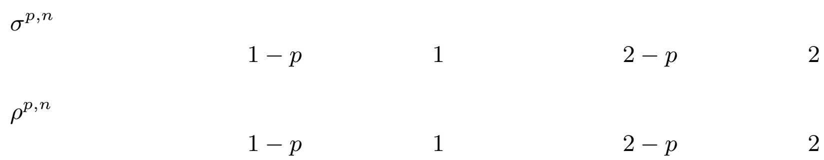
We now extend the results for the case of finite words. As before we replace the infinite words $\alpha^{p,n}$ and $\beta^{p,n}$ by the sequences $\langle \sigma_i^{p,n} \rangle$ and $\langle \rho_i^{p,n} \rangle$, respectively, which are given by $\sigma_i^{p,n} = \mu_1 \tau^i$, and $\rho_i^{p,n} = \mu_2 \tau^i$, where $\mu_1 = (b, d)(b, 2d) \cdots (b, (n+2)d)(a, (n+3)d)$, $\mu_2 = (b, d)(b, 2d) \cdots (b, (n+1)d)(a, (n+2)d)$ and $\tau = (b, d)(b, 2d) \cdots (b, (n+3)d)(a, (n+4)d)$. Further, we replace $\eta^{p,n}$ by $\langle {\sigma'}_i^{p,n} \rangle$ where ${\sigma'}_i^{p,n} = \tau^i$. We can now mimic the proof sketched above by replacing satisfiability by ultimate satisfiability of the continuous semantics.

7 Continuous semantics is strictly more expressive

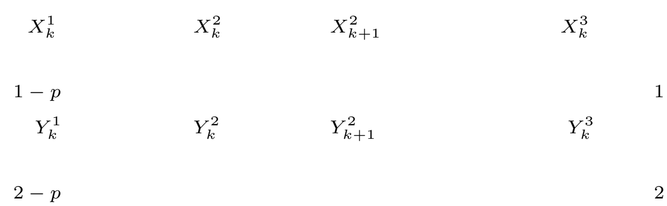
In this section we show that the language L_{2ins} is not expressible by $\mathrm{MTL}^{pw}_{S_I}$ but is expressible by MTL^c. This leads to the strict containment of the pointwise versions of the logics in their corresponding continuous versions, since L_{2ins} is not expressible by $\mathrm{MTL}^{pw}_{S_I}$, and hence is not expressible by MTL^{pw}_{S} and MTL^{pw}, but is expressible by MTL^c, and hence by MTL^c_S and $\mathrm{MTL}^c_{S_I}$.

We will first show the result for finite words and then show how it can be extended for infinite words. L_{2ins} is the timed language over $\Sigma = \{a, b\}$ such that every timed word in the language contains two consecutive a's such that there exist two time points between their times of occurrences, at distance one in the future from each of which there is an a. Formally, $L_{2ins} = \{\sigma \in T\Sigma^* \mid \sigma = (a_1, t_1)\cdots(a_n, t_n), \exists i \in \mathbb{N} : a_i = a_{i+1} = a, \exists t_j, t_k \in (t_i + 1, t_{i+1} + 1) : j \neq k, a_j = a_k = a\}$.

Let $p \in \mathbb{Q}_{>0}$, where $p = 1/k$ and $k \in \mathbb{N}$, and let $n \in \mathbb{N}$. Let $d = p/(2n + 3)$. We give the two models $\sigma^{p,n}$ and $\rho^{p,n}$ which are as defined follows. $\sigma^{p,n}$ is given by $(a, 1 - p + d/2)(a, 1 - p + 3d/2)\cdots(a, 1 - p/2 - d)(a, 1 - p/2)(a, 1 - p/2 + d)\cdots(a, 1 - d/2)(a, 2 - p + d)(a, 2 - p + 2d)\cdots(a, 2 - d)$ and $\rho^{p,n}$ is given by $(a, 1 - p + d/2)(a, 1 - p + 3d/2)\cdots(a, 1 - p/2 - d)(a, 1 - p/2 + d)\cdots(a, 1 - d/2)(a, 2 - p + d)(a, 2 - p + 2d)\cdots(a, 2 - d)$. They are depicted below.



It is easy to see that $\sigma^{p,n} \notin L_{2ins}$ and $\rho^{p,n} \in L_{2ins}$. We use the following lemmas to show that no MTL_{S_I} formula can define L_{2ins} in the pointwise semantics. Let $X^2_k = \{k+1, \cdots, 2n+3-k\}$ and $Y^2_k = \{2n+3+(k+1), \cdots, 2n+3+(2n+2-k)\}$.



Lemma 4. *Let $k \in \mathbb{N}$ and $0 \leq k \leq n$. Let $\varphi \in \mathrm{MTL}_{S_I}(p, k)$. Then for all $i, j \in X^2_k$, $\sigma^{p,n}, i \models_{pw} \varphi$ iff $\sigma^{p,n}, j \models_{pw} \varphi$ and for all $i, j \in Y^2_k$, $\sigma^{p,n}, i \models_{pw} \varphi$ iff $\sigma^{p,n}, j \models_{pw} \varphi$.* □

Given an $n \in \mathbb{N}$, we define a partial function $h_n : \mathbb{N} \to \mathbb{N}$ which is defined for all $i \in \mathbb{N}$ except for $n+2$. $h_n(i) = i$ if $i < n+2$ and $h_n(i) = i-1$ if $i > n+2$. $h_n(i)$ is the position in $\rho^{p,n}$ corresponding to the position i in $\sigma^{p,n}$ in the sense that the time of the $h_n(i)$-th action in $\rho^{p,n}$ is the same as that of the i-th action in $\sigma^{p,n}$ (hence it is not defined for $n+2$).

Lemma 5. *Let* $k \in \mathbb{N}$, $0 \leq k \leq n$ *and let* $\varphi \in \mathrm{MTL}_{S_I}(p,k)$*. For all* $i,j \in X_k^2 - \{n+2\}$, $\rho^{p,n}, h_n(i) \models_{pw} \varphi$ *iff* $\rho^{p,n}, h_n(j) \models_{pw} \varphi$ *and for all* $i,j \in Y_k^2$, $\rho^{p,n}, h_n(i) \models_{pw} \varphi$ *iff* $\rho^{p,n}, h_n(j) \models_{pw} \varphi$*.* □

Corollary 2. *Let* $\varphi \in \mathrm{MTL}_{S_I}(p,n)$*. Then* $\sigma^{p,n}, n{+}1 \models_{pw} \varphi$ *iff* $\sigma^{p,n}, n{+}2 \models_{pw} \varphi$ *iff* $\sigma^{p,n}, n+3 \models_{pw} \varphi$ *and* $\rho^{p,n}, h_n(n+1) \models_{pw} \varphi$ *iff* $\rho^{p,n}, h_n(n+3) \models_{pw} \varphi$*.* □

Theorem 8. *For any* $\varphi \in \mathrm{MTL}_{S_I}(p,n)$ *and* $i \in \mathbb{N}$*, where* $i \neq n+2$, $\sigma^{p,n}, i \models_{pw} \varphi$ *iff* $\rho^{p,n}, h_n(i) \models_{pw} \varphi$*.* □

Corollary 3. *For any* $\varphi \in \mathrm{MTL}_{S_I}(p,n)$, $\sigma^{p,n}, 0 \models_{pw} \varphi$ *iff* $\rho^{p,n}, 0 \models_{pw} \varphi$*.* □

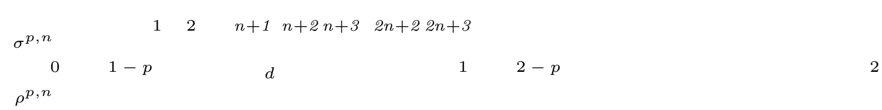
It can now be argued that no MTL_{S_I} formula in the pointwise semantics can define L_{2ins}. But the following MTL formula in the continuous semantics defines L_{2ins}. $\Diamond(a \wedge \neg\varphi_{act} U(\neg\varphi_{act} \wedge \Diamond_{[1,1]} a \wedge \neg\varphi_{act} U(\neg\varphi_{act} \wedge \Diamond_{[1,1]} a \wedge \neg\varphi_{act} U a)))$.

The above result can be extended for infinite timed words by replacing the finite models above, by infinite models which are similar to their counterparts in the interval $[0,2]$ but contain a b at every integer time greater than 2. Then the proof for infinite models is very similar to that of the finite models.

8 Strict containment of MTL_S^{pw} in $\mathrm{MTL}_{S_I}^{pw}$

In this section we show the strict containment of MTL_S^{pw} in $\mathrm{MTL}_{S_I}^{pw}$ by showing that the language L_{ni} is not expressible by MTL_S^{pw}, but is expressible by $\mathrm{MTL}_{S_I}^{pw}$.

The timed language L_{em} (for "exact match") consists of timed words in which for every b in the interval $(0,1)$, there is a b in the future which is at time distance 1 from it, and for every b in the interval $(1,2)$, there is a b in the past which is at time distance 1 from it. The structure of the proof is very similar to that of the previous section. We give below a description of the models.

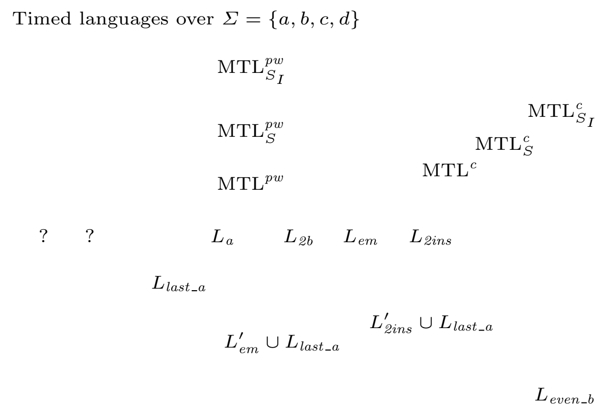


Let $x = 1-p$, $y = 2-p$ and $d = p/(2n+4)$. Then $\sigma^{p,n}$ is given by $(b, x+d)(b, x+2d)\cdots(b, x+(n+1)d)(b, x+(n+2)d)(b, x+(n+3)d)\cdots(b, x+(2n+2)d)(b, x+(2n+3)d)(b, y+d)(b, y+2d)\cdots(b, y+(n+1)d)(b, y+(n+2)d)(b, y+(n+3)d)\cdots(b, y+(2n+2)d)(b, y+(2n+3)d)$.
And $\rho^{p,n}$ is given by $(b, x{+}d)(b, x{+}2d)\cdots(b, x{+}(n{+}1)d)(b, x{+}(n{+}3)d)\cdots(b, x+(2n+2)d)(b, x+(2n+3)d)(b, y+d)(b, y+2d)\cdots(b, y+(n+1)d)(b, y+(n+2)d)(b, y+(n+3)d)\cdots(b, y+(2n+2)d)(b, y+(2n+3)d)$.

Taking $X_k^2 = \{k+1, \cdots, 2n+3-k\}$ and $Y_k^2 = \{(2n+3)+k+1, \cdots, (2n+3)+2n+3-k\}$, we can follow the proof of the inexpressibility of L_{2ins}. The proof can be similarly extended for the case of infinite words.

9 Expressiveness of MTL with past operators

In the diagram below, we summarize the relative expressiveness results. The timed language L_a consists of timed words containing an a. L'_{em} and L'_{2ins} are obtained from L_{em} and L_{2ins} respectively by replacing the occurrences of a's and b's in the timed words by c's and d's, respectively.



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