BCU Mathematics Contest 1999

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BCU MATHEMATICS CONTEST, MARCH 13, 1999

SOLUTIONS TO THE TEAM CONTEST PAPER

1. Since a non-leap year has 365 days, we're required to find positive integers a, b such that
$$ab + a + b = 365.$$
Now $ab + a + b = (a+1)(b+1) - 1$, hence a, b are positive integral solutions of the equation
$$(a+1)(b+1) = 366 = 1.2.3.61.$$
Since a, b are positive, and the equation is symmetric in a, b the only possibile solution pairs are $(1, 182), (2, 121), (5, 60)$. The answer is therefore 3.

   Contestants who find these solutions by trial and erorr, and do not show that these are the only solutions, get a maximum of 4 points.

2. If
$$a_n = \frac{A}{n-1} + \frac{B}{n+1}, \; n = 2, 3, \ldots,$$
then, in particular,
$$a_2 = A + \frac{1}{3}B, \qquad a_3 = \frac{1}{2}A + \frac{1}{4}B,$$
i.e.,
$$\frac{2}{3} = A + \frac{1}{3}B, \qquad \frac{1}{4} = \frac{1}{2}A + \frac{1}{4}B,$$
whence
$$2 = 3A + B, \qquad 1 = 2A + B.$$

Thus $A = 1$ and $B = -1$. It remains to show that these values work for all $n \geq 2$. But this is easy: if $n \geq 2$, then

$$\begin{aligned} \frac{1}{n-1} + \frac{-1}{n+1} &= \frac{n+1-(n-1)}{(n-1)(n+1)} \\ &= \frac{2}{n^2-1} \\ &= a_n. \end{aligned}$$

OR

$$\begin{aligned} \frac{2}{n^2-1} &= \frac{2}{(n-1)(n+1)} \\ &= \frac{A}{n-1} + \frac{B}{n+1} \\ &= \frac{A(n+1)+B(n-1)}{(n-1)(n+1)} \\ &= \frac{(A+B)n+(A-B)}{n^2-1}, \end{aligned}$$

for $n \geq 2$ iff $2 = (A+B)n + A - B$, $n = 2, 3, \ldots$. This is so iff $A = -B = 1$.

We now have for $k = 2, 3, \ldots$

$$\begin{aligned} s_k &= \sum_{n=2}^{k} a_n \\ &= \sum_{n=2}^{k} \left[\frac{1}{n-1} - \frac{1}{n+1}\right] \\ &= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots \\ &+ \left(\frac{1}{k-2} - \frac{1}{k}\right) + \left(\frac{1}{k-1} - \frac{1}{k+1}\right). \end{aligned}$$

Removing brackets and cancelling we see that

$$s_k = 1 + \frac{1}{2} - \frac{1}{k} - \frac{1}{k+1} = \frac{3}{2} - \frac{2k+1}{k(k+1)}.$$

OR

$$\begin{aligned} s_k = \sum_{nk=2}^{k} a_n &= \sum_{n=2}^{k} \left[\frac{1}{n-1} - \frac{1}{n+1}\right] \\ &= \sum_{n=2}^{k} \left[\left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)\right] \\ &= \sum_{n=2}^{k} \left(\frac{1}{n-1} - \frac{1}{n}\right) + \sum_{n=2}^{k} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{k} + \frac{1}{2} - \frac{1}{k+1} \\ &= \frac{3}{2} - \frac{2k+1}{k(k+1)}. \end{aligned}$$

OR

$$\begin{aligned} s_k = \sum_{nk=2}^{k} a_n &= \sum_{n=2}^{k} \left[\frac{1}{n-1} - \frac{1}{n+1}\right] \\ &= \sum_{n=2}^{k} \frac{1}{n-1} - \sum_{n=2}^{k} \frac{1}{n+1} \\ &= \sum_{n=1}^{k-1} \frac{1}{n} - \sum_{n=3}^{k+1} \frac{1}{n} \\ &= 1 + \frac{1}{2} - \frac{1}{k} - \frac{1}{k+1} \\ &= \frac{3}{2} - \frac{2k+1}{k(k+1)}. \end{aligned}$$

Clearly, then

$$s_{100} - s_{10} = \frac{21}{10.11} - \frac{201}{100.101} < \frac{21}{10.11} < \frac{1}{5}.$$

OR

$$s_{100} - s_{10} = \frac{1}{10} + \frac{1}{11} - \left(\frac{1}{100} + \frac{1}{101}\right) < \frac{2}{10} = \frac{1}{5},$$

and

$$s_{100} - s_{10} > \frac{2}{11} - \frac{2}{100} = \frac{2.89}{11.100} > \frac{4}{25}.$$

Thus the error lies between 4/25 and 1/5.

3.

$$y' = 3x^2 - 12x + 12 = 3(x^2 - 4x + 4) = 3(x-2)^2 \geq 0, \ \forall x.$$

Hence y is increasing on $(-\infty, \infty)$. Also,

$$y'' = 6x - 12x = 6(x-2).$$

Clearly, $y'' \geq 0$, $\forall x \geq 2$ and $y'' \leq 0$, $\forall x \leq 2$. Hence y is convex if $x \geq 2$ and concave if $x \leq 2$. It has no local extrema and one point of inflection at $x = 2$. Also,

$$\lim_{x\to\infty} = \infty, \ \lim_{x\to-\infty} = -\infty.$$

Since y is a cubic polynomial with real coefficients it crosses the x-axis at least once. To see where, note that

$$y = x^3 - 6x^2 + 12x - 7 = (x-2)^3 + 1 = 0$$

if $x = 1$. (Alternatively, the root can be found by trial and error.) Then

$$y = (x-1)(x^2 - 5x + 7) = (x-1)((x-5/2)^2 + \frac{3}{4}).$$

Hence y crosses the horizontal axis precisely once.

OR

It is $x^3 - 6x^2 + 12x - 7 = x^3 - 3 \cdot 2x^2 + 3 \cdot 2^2 x - 2^3 + 1 = (x-2)^3 + 1$. Therefore, the function has the equation

$$y = (x-2)^3 + 1,$$

i.e. it is the cubic funcion $y = x^3$ where the point of inflection is moved to $(2, 1)$. Hence, it is convex if $x \geq 2$, concave if $x \leq 2$, and has point of inflection $(2, 1)$.

4. Determine the area of the bounded region enclosed by the set of points in the xy-plane that satisfy the equation

$$xy + y^2 - yx^2 - x^3 = 0.$$

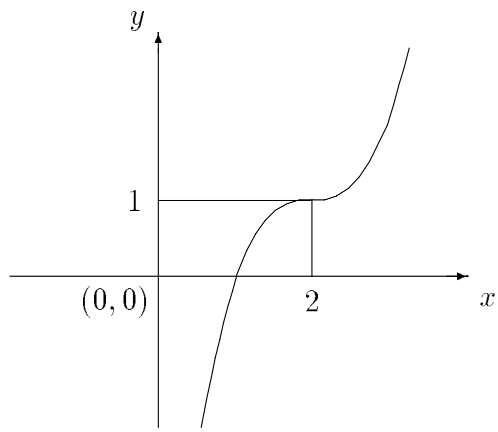

Figure 1: Graph of $y = x^3 - 6x^2 + 12x - 7$

Note that $xy + y^2 - yx^2 - x^3 = (y+x)(y-x^2)$. Thus (x, y) belongs to the set

$$\{(x,y) : xy + y^2 - yx^2 - x^3 = 0\}$$

iff it lies on the line $y = -x$ or on the parabola $y = x^2$. The *bounded* region enclosed by these curves is therefore the set

$$\{(x,y) : x^2 \le y \le -x, -1 \le x \le 0\},$$

the area of which is

$$\int_{-1}^{0} (-x)\, dx - \int_{-1}^{o} x^2\, dx = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

OR

Not everybody will spot the factorization. Some may treat $xy + y^2 - yx^2 - x^3 = 0$ as a quadratic in y—$y^2 + (x - x^2)y - x^3 = 0$—and solve for y:

$$\begin{aligned} y &= \frac{-(x-x^2) \pm \sqrt{(x-x^2)^2 + 4x^3}}{2} \\ &= \frac{-(x-x^2) \pm \sqrt{x^2 + 2x^3 + x^4}}{2} \\ &= \frac{-(x-x^2) \pm |x + x^2|}{2}. \end{aligned}$$

Now denoting the roots by y_+, y_-we see that

$$y_+ = \begin{cases} x^2, & \text{if } x(1+x) \geq 0 \\ -x, & \text{if } x(1+x) \leq 0, \end{cases}$$

and

$$y_- = \begin{cases} -x, & \text{if } x(1+x) \geq 0 \\ x^2, & \text{if } x(1+x) \leq 0, \end{cases}$$

It follows that the set

$$\{(x, y) : xy + y^2 - yx^2 - x^3 = 0\}$$

is the union of the line $y = -x$ and the parabola $y = x^2$. The problem can be finished as before.

Inevitably, some of those who treat it this way will say that $\sqrt{(x + x^2)^2} = x + x^2$ and simplify the work for themselves! But they should be penalised!

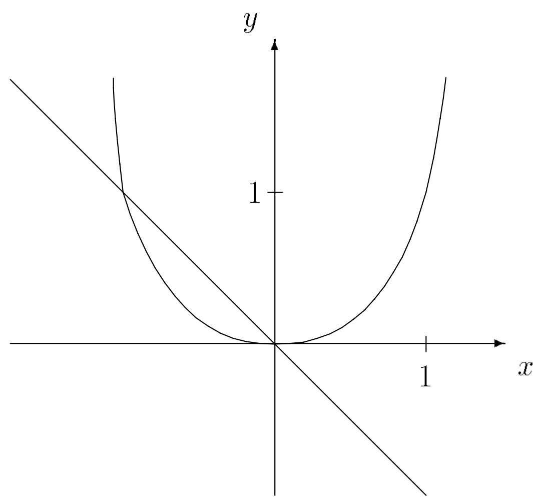

Figure 2: The two graphs enclosing the area in question.

BCU MATHEMATICS CONTEST, MARCH 13, 1999

SOLUTIONS TO THE INDIVIDUAL CONTEST PAPER

1. Picture:

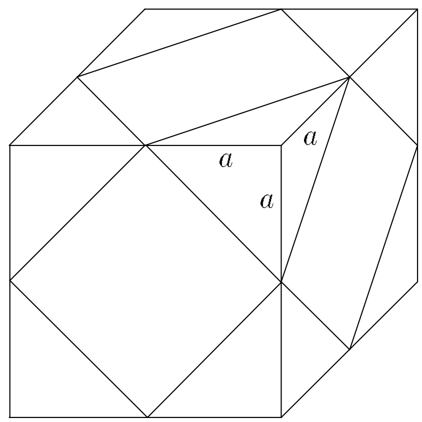

Figure 3: The cuts on the wooden cube.

Volume:

Any of the pyramids which is cut away at some edge has volume $V_P = \frac{a^3}{6}$. The volume of the remaining solid is therefore

$$V_R = 8a^3 - \frac{4}{3}a^3 = \frac{20}{3}a^3.$$

Surface:

Along the cuts appear equilateral triangles with sides of length $a\sqrt{2}$. Hence, each of those triangles has area $A_T = \frac{a^2}{2}\sqrt{3}$. The surface area of the remaining solid is therefore

$$O_R = 24a^2 - 8\cdot 3\cdot \frac{a^2}{2} + 8\cdot\frac{a^2}{2}\sqrt{3} = 12a^2 + 4a^2\sqrt{3}.$$

2. The largest root of the equation is

$$\beta = \frac{-b+\sqrt{b^2-4ac}}{2a}.$$

For $a \to 0$, both numerator and denominator tend to 0.

First solution:

$$\beta = \frac{(\sqrt{b^2-4ac}-b)(\sqrt{b^2-4ac}+b)}{2a(\sqrt{b^2-4ac}+b)} = \frac{-2c}{\sqrt{b^2-4ac}+b}$$

and therefore

$$\lim_{a\to 0}\beta = -\frac{c}{b}.$$

Second solution:

Using de l'Hospital's rule we obtain

$$\lim_{a\to 0}\beta = \lim_{a\to 0}\frac{\frac{-2c}{\sqrt{b^2-4ac}}}{2} = -\frac{c}{b}.$$

Third Solution:

Let $h = -4ac$ and, for $x > 0$, define $f(x) = \sqrt{x}$. Then $b = f(b^2)$ and

$$\beta = \frac{-b+\sqrt{b^2-4ac}}{2a} = -2c\left(\frac{f(b^2+h)-f(b^2)}{h}\right).$$

Now as $a \to 0$, $h \to 0$ and so

$$\begin{aligned}
\lim_{a\to 0}\beta &= -2c\lim_{h\to 0}\frac{f(b^2+h)-f(b^2)}{h}\\
&= -2cf'(b^2)\\
&= -2c\frac{1}{2\sqrt{b^2}}\\
&= -\frac{c}{b},
\end{aligned}$$

since

$$f'(x) = \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}, \ \forall x > 0.$$

Comment. All three proofs rely on the crucial fact that f is continuous on $[0, \infty)$.

3. First Solution. Join B to E and to Y. Let $\theta = \angle YAX$ and $\phi = \angle ACX$, so that $\phi = \pi/2 - \theta$. Then since $\angle BXA = \angle BYA = \pi/2$ we see that $\angle CBY = \theta$. But the triangles BCY and BYE are similar. Hence $\angle YBE = \angle CBY = \theta$ and $\angle BEC = \phi$. It now follows that

$$\angle ABE = \angle ABC + \angle CBY + \angle YBE = \phi - 2\theta + \theta + \theta = \phi.$$

Hence the triangle AEB is isosceles and so $|AE| = |AB| = d$.

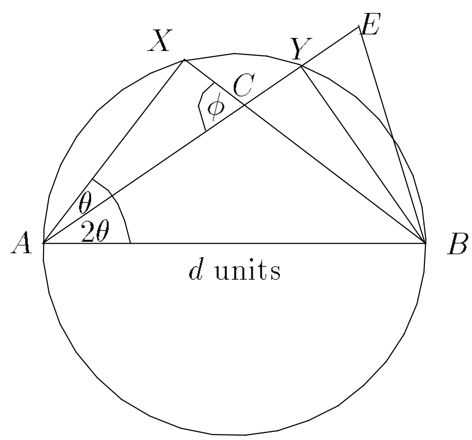

Figure 4: Picture for Question 3.

Second Solution. Let $x = |AC|, y = |CY|$. Then

$$x \cos\theta = |AX| = d \cos \angle BAX = d \cos 3\theta,$$

and

$$x + y = |AY| = d \cos \angle BAY = d \cos 2\theta.$$

Hence

$$y = d[\cos 2\theta - \frac{\cos 3\theta}{\cos\theta}] = \frac{d(\cos 2\theta\cos\theta - \cos 3\theta)}{\cos\theta},$$

and so

$$\begin{aligned}
|AE| &= x + 2y \\
&= \frac{d(2\cos 2\theta\cos\theta - \cos 3\theta)}{\cos\theta} \\
&= \frac{d(2\cos 2\theta\cos\theta - \cos 2\theta\cos\theta + \sin 2\theta\sin\theta)}{\cos\theta} \\
&= \frac{d(\cos 2\theta\cos\theta + \sin 2\theta\sin\theta)}{\cos\theta} \\
&= \frac{d\cos\theta}{\cos\theta} \\
&= d.
\end{aligned}$$

BCU MATHEMATICS CONTEST, MARCH 13, 1999

SOLUTIONS FOR THE SPEED CONTEST PAPER FOR TEAMS

1. Let a be the number of initial participants. Then, initially, we have

$$\frac{55}{100} \cdot a$$

girls among them. We therefore obtain the equation

$$\frac{60}{100}(a+5) = \frac{55}{100}a + 5$$

and therefore

$$a = 40.$$

So, initially, there were 40 participants and $\frac{55 \cdot 40}{100} = 22$ girls among them.

2. For all pairs (x, y) with $x^2 + y^2 = 1$ we obtain

$$f(x,y) = 2(x^2 + y^2) + y^2 = 2 + y^2.$$

Since the condition on x and y implies that $0 \leq y^2 \leq 1$, we have

$$2 \leq f(x,y) \leq 3.$$

We obtain the maximum value 3 for $y = -1$ and $y = 1$, and the minimum value 2 for $y = 0$. The solutions therefore are:

Maxima: $(0, -1)$, $(0, 1)$

Minima: $(1, 0)$, $(-1, 0)$

3. Since $1 \leq b < a$, we have $b \leq b^2 < a^2$ and so

$$b - a^2 < b - b^2 < b < a < 2a$$

which suffices.

OR

$$b - a^2 < b - b^2 \leq b - b = 0 < 1 \leq a < 2a.$$

4. The height h of the triangle is

$$h = \sqrt{a^2 - \frac{1}{4}a^2} = \frac{a}{2}\sqrt{3}.$$

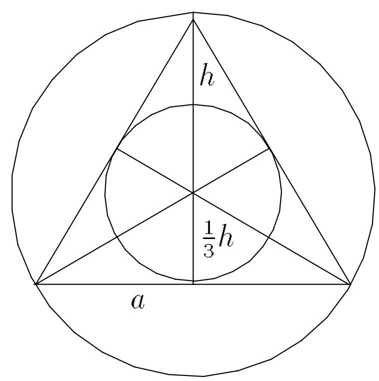

Figure 5: Question 4

Since the altitudes of the triangle intersect at a third of their lengths, and the centres of the circles coincide at the point of intersection of the altitudes, we get

$$r_i = \frac{a}{6}\sqrt{3}$$

as the radius of the inscribed circle, and

$$r_c = \frac{a}{3}\sqrt{3}$$

as the radius of the circumscribed circle.

5. For $x = 0$ the recursion formula reduces to

$$p_{n+2}(0) = p_{n+1}(0).$$

Hence we obtain

$$\begin{aligned} p_1(0) &= 1, \\ p_2(0) &= 3, \\ p_3(0) &= 3, \\ p_5(0) &= p_4(0) = p_3(0) = 3. \end{aligned}$$

For $x = -1$ the recursion formula is

$$p_{n+2}(-1) = p_{n+1}(-1) - 2p_n(-1),$$

and therefore

$$\begin{aligned} p_1(-1) &= 1, \\ p_2(-1) &= 2, \\ p_3(-1) &= 0, \\ p_4(-1) &= -4, \\ p_5(-1) &= -4. \end{aligned}$$

6. We obtain

$$\begin{aligned} 7^1 &= \ldots 07 \\ 7^2 &= \ldots 49 \\ 7^3 &= \ldots 43 \\ 7^4 &= \ldots 01 \\ 7^5 &= \ldots 07 \\ 7^6 &= \ldots 49 \\ &\vdots \\ 7^{76} &= \ldots 01 \end{aligned}$$

since $76 \bmod 4 = 0$. Hence, the last two digits are 01.

7. Each of the 27 students taking three courses takes a course in Mathematics or Computer Science. Since 34 students take Spanish or History, $45 - 34 = 11$ students take Mathematics or Computer Science, but neither of Spanish or History. Each of the latter group of students therefore attends a maximum of two courses. From this we obtain a lower bound of $27 + 11 = 38$ students taking Mathematics or Computer Science.

   We show that the bound is sharp, by giving a distribution of students among the courses such that exactly 38 students take Mathematics or Computer Science. It is as follows:

   27 students take 3 courses.
   11 students take Mathematics and Computer Science only.
   7 students take Spanish and History only.

   With this distribution, the assumptions of the question are satisfied and 38 students take Mathematics or Computer Science. Hence, the bound is sharp.