Paraconsistent Reasoning for OWL 2*

Yue Ma[1] and Pascal Hitzler[2]

[1]Institute LIPN, Université Paris-Nord (LIPN - UMR 7030), France
[2]Institute AIFB, Universität Karlsruhe, Germany
yue.ma@lipn.univ-paris13.fr, pascal@pascal-hitzler.de

Abstract. A four-valued description logic has been proposed to reason with description logic based inconsistent knowledge bases. This approach has a distinct advantage that it can be implemented by invoking classical reasoners to keep the same complexity as under the classical semantics. However, this approach has so far only been studied for the basid description logic $\mathcal{ALC}$. In this paper, we further study how to extend the four-valued semantics to the more expressive description logic $\mathcal{SROIQ}$ which underlies the forthcoming revision of the Web Ontology Language, OWL 2, and also investigate how it fares when adapated to tractable description logics including $\mathcal{EL}$++, DL-Lite, and Horn-DLs. We define the four-valued semantics along the same lines as for $\mathcal{ALC}$ and show that we can retain most of the desired properties.

1 Introduction

Expressive and tractable description logics have been well-studied in the field of semantic web methods and applications, see e.g. [22, 6]. In particular, description logics are the foundations of the Web Ontology Language OWL [7, 17] and its forthcoming revision, OWL 2 [24]. However, real knowledge bases and data for Semantic Web applications will rarely be perfect. They will be distributed and multi-authored. They will be assembled from different sources and reused. It is unreasonable to expect such realistic knowledge bases to be always logically consistent, and it is therefore important to study ways of dealing with inconsistencies in both expressive and tractable description logic based ontologies, as classical description logics break down in the presence of inconsistent knowledge.

About inconsistency handling of ontologies based on description logics, two fundamentally different approaches can be distinguished. The first is based on the assumption that inconsistencies indicate erroneous data which is to be repaired in order to obtain a consistent knowledge base, e.g. by selecting consistent subsets for the reasoning process [21, 8, 5]. The other approach yields to the insight that inconsistencies are a natural phenomenon in realistic data which are to be handled by a logic which tolerates it [20, 23, 13]. Such logics are called paraconsistent, and the most prominent of them are based on the use of additional truth values standing for *underdefined* (i.e. neither true nor false) and *overdefined* (or *contradictory*, i.e. both true and false). Such logics are appropriately

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called *four-valued logics* [2, 1]. We believe that either of the approaches is useful, depending on the application scenario. Besides this, four-valued semantics proves useful for measuring inconsistency of ontologies [16], which can provide context information for facilitating inconsistency handling.

In this paper, we extend our study of paraconsistent semantics for $\mathcal{ALC}$ from [13]. This approach has the pleasing two properties that (1) reasoning under the paraconsistent semantics can be reduced to reasoning under classical semantics and (2) the transformations required for the reduction from the paraconsistent semantics to the classical semantics is linear in the size of the knowledge base. In this paper, we will carry these results over to $\mathcal{SROIQ}$, which underlies OWL 2, and also study its impact for several tractable description logics around OWL 2. We also present a slight modification to the semantics presented in [13]. In more detail, the contributions of the paper are as follows.

- The extension of four-valued semantics to $\mathcal{SROIQ}$ is defined. Specially, we show that it still can be reduced to classical semantics regardless its high expressivity.
- The four-valued semantics is studied for the tractable description logics $\mathcal{EL}$++, Horn-DLs, and DL-Lite, for some of these adaptations of the semantics are made. We show that under certain restrictions our approach retains tractability.
- Compared with our existing work on four-valued semantics for $\mathcal{ALC}$, in this paper, we do not impose four-valued semantics on roles (with the exception of DL-Lite). The reasons are: (1) Negative roles are not used as concept constructors in $\mathcal{ALC}$, $\mathcal{SROIQ}$, $\mathcal{EL}$++, or Horn-DLs[1] such that contradiction caused directly by roles can safely be ignored. (2) This modified four-valued semantics is more similar to the classical semantics. (3) Four-valued semantics is semantically weaker than the classical semantics which means that there are undesired missing conclusions under the semantics from [13], which is not the case in our modified approach.

The paper is structured as follows. We first review briefly the four-valued semantics for $\mathcal{ALC}$ in Section 2. Then we study the four-valued semantics for expressive description logics in Section 3 and four-valued semantics for tractable description logics in Section 4, respectively. We conclude and discuss future work in Section 5.

This paper is an extension and revision of the workshop paper [14].

2 Preliminaries

2.1 The Four-valued Semantics for $\mathcal{ALC}$ – With a Slight Modification

We describe the syntax and semantics of four-valued description logic $\mathcal{ALC}4$ from [13] with a slight modification. Syntactically, $\mathcal{ALC}4$ hardly differs from $\mathcal{ALC}$. Complex concepts and assertions are defined in exactly the same way. For class inclusion axioms, however, significant effort has been devoted on the intuitions behind these different implications in [13]. For the four-valued semantics, three different kinds of inclusions can

[1] Note that OWL 2 allows negative property assertions like $R(a, b)$ in the ABox; however they can be considered syntactic sugar on top of $\mathcal{SROIQ}$ since they can be written as $\{a\} \sqsubseteq \forall R.\neg\{b\}$. In this paper, we assume that all negative property assertions have been rewritten in this way.

be used, as follows. They serve different underlying intuitions and differ in inferential strength. A detailed discussion of this has been presented in [13] which we will not repeat here.

$$C \mapsto D \text{ (material inclusion axiom),}$$
$$C \sqsubset D \text{ (internal inclusion axiom),}$$
$$C \rightarrow D \text{ (strong inclusion axiom).}$$

Semantically, interpretations map individuals to elements of the domain of the interpretation, as usual. For concepts, however, modifications are made to the notion of interpretation in order to allow for reasoning with inconsistencies.

Intuitively, in four-valued logic we need to consider four situations which can occur in terms of containment of an individual in a concept: (1) we know it is contained, (2) we know it is not contained, (3) we have no knowledge whether or not the individual is contained, (4) we have contradictory information, namely that the individual is both contained in the concept and not contained in the concept. There are several equivalent ways how this intuition can be formalized, one of which is described in the following.

For a given domain Δ^I and a concept C, an interpretation over Δ^I assigns to C a pair $\langle P, N\rangle$ of (not necessarily disjoint) subsets of Δ^I. Intuitively, P is the set of elements known to belong to the extension of C, while N is the set of elements known to be not contained in the extension of C. For simplicity of notation, we define functions $proj^+(\cdot)$ and $proj^-(\cdot)$ by $proj^+\langle P, N\rangle = P$ and $proj^-\langle P, N\rangle = N$.

Formally, a four-valued interpretation is a pair $I = (\Delta^I, \cdot^I)$ with Δ^I as domain, where $\cdot^I$ is a function assigning elements of Δ^I to individuals, and subsets of $(\Delta^I)^2$ to concepts, such that the conditions in Table 1 are satisfied. Note that the semantics of roles here remains unchanged from the classical two-valued case, and in this point the semantics presented here differs from that in [13]. Intuitively, inconsistencies always arise on concepts, and not on roles, at least in the absence of role negation, which is often assumed when studying DLs. We will see in this paper that this approach can be used to tolerate inconsistency, not only for $\mathcal{ALC}$ but also for more expressive description logics. This is an improvement over [13] in the sense that we would like to make as few changes as possible when extending the classical semantics to a four-valued semantics for handling inconsistency.

The semantics of the three different types of inclusion axioms is formally defined in Table 2 (together with the semantics of concept assertions). Again we refer to the discussions in [13] for details.

We say that a four-valued interpretation (a *4-interpretation*) I satisfies a four-valued knowledge base $\mathcal{O}$ (i.e. is a model, or *4-model*, of it) iff it satisfies each assertion and each inclusion axiom in $\mathcal{O}$. A knowledge base $\mathcal{O}$ is satisfiable (unsatisfiable) iff there exists (does not exist) such a model.

2.2 Reduction from Four-valued Semantics of $\mathcal{ALC}$ to Classical Semantics

It is a pleasing property of $\mathcal{ALC}4$, that it can be translated easily into classical $\mathcal{ALC}$, such that paraconsistent reasoning can be simulated by using standard $\mathcal{ALC}$ reasoning algorithms.

Table 1. Semantics of $\mathcal{ALC}4$ Concepts

Constructor Syntax	Semantics
A	$A^I = \langle P, N \rangle$, where $P, N \subseteq \Delta^I$
R	$R^I \subseteq \Delta^I \times \Delta^I$
o	$o^I \in \Delta^I$
$\top$	$\langle \Delta^I, \emptyset \rangle$
$\bot$	$\langle \emptyset, \Delta^I \rangle$
$C_1 \sqcap C_2$	$\langle P_1 \cap P_2, N_1 \cup N_2 \rangle$, if $C_i^I = \langle P_i, N_i \rangle$ for $i = 1, 2$
$C_1 \sqcup C_2$	$\langle P_1 \cup P_2, N_1 \cap N_2 \rangle$, if $C_i^I = \langle P_i, N_i \rangle$ for $i = 1, 2$
$\neg C$	$(\neg C)^I = \langle N, P \rangle$, if $C^I = \langle P, N \rangle$
$\exists R.C$	$\langle \{x \mid \exists y, (x, y) \in R^I$ and $y \in \mathit{proj}^+(C^I)\}$, $\{x \mid \forall y, (x, y) \in R^I$ implies $y \in \mathit{proj}^-(C^I)\} \rangle$
$\forall R.C$	$\langle \{x \mid \forall y, (x, y) \in R^I$ implies $y \in \mathit{proj}^+(C^I)\}$, $\{x \mid \exists y, (x, y) \in R^I$ and $y \in \mathit{proj}^-(C^I)\} \rangle$

Table 2. Semantics of inclusion axioms in $\mathcal{ALC}4$

Axiom Name	Syntax	Semantics
material inclusion	$C_1 \mapsto C_2$	$\Delta^I \setminus \mathit{proj}^-(C_1^I) \subseteq \mathit{proj}^+(C_2^I)$
internal inclusion	$C_1 \sqsubset C_2$	$\mathit{proj}^+(C_1^I) \subseteq \mathit{proj}^+(C_2^I)$
strong inclusion	$C_1 \rightarrow C_2$	$\mathit{proj}^+(C_1^I) \subseteq \mathit{proj}^+(C_2^I)$ and $\mathit{proj}^-(C_2^I) \subseteq \mathit{proj}^-(C_1^I)$
individual assertions	$C(a)$	$a^I \in \mathit{proj}^+(C^I)$
	$R(a, b)$	$(a^I, b^I) \in R^I$

Definition 1 *(Concept transformation) For any given concept C, its transformation $\pi(C)$ is the concept obtained from C by the following inductively defined transformation.*

- *If $C = A$ for A an atomic concept, then $\pi(C) = A^+$, where A^+ is a new concept;*
- *If $C = \neg A$ for A an atomic concept, then $\pi(C) = A'$, where A' is a new concept;*
- *If $C = \top$, then $\pi(C) = \top$;*
- *If $C = \bot$, then $\pi(C) = \bot$;*
- *If $C = E \sqcap D$ for concepts D, E, then $\pi(C) = \pi(E) \sqcap \pi(D)$;*
- *If $C = E \sqcup D$ for concepts D, E, then $\pi(C) = \pi(E) \sqcup \pi(D)$;*
- *If $C = \exists R.D$ for D a concept and R is a role, then $\pi(C) = \exists R.\pi(D)$;*
- *If $C = \forall R.D$ for D a concept and R is a role, then $\pi(C) = \forall R.\pi(D)$;*
- *If $C = \neg\neg D$ for a concept D, then $\pi(C) = \pi(D)$;*
- *If $C = \neg(E \sqcap D)$ for concepts D, E, then $\pi(C) = \pi(\neg E) \sqcup \pi(\neg D)$;*
- *If $C = \neg(E \sqcup D)$ for concepts D, E, then $\pi(C) = \pi(\neg E) \sqcap \pi(\neg D)$;*
- *If $C = \neg(\exists R.D)$ for D a concept and R is a role, then $\pi(C) = \forall R.\pi(\neg D)$;*
- *If $C = \neg(\forall R.D)$ for D a concept and R is a role, then $\pi(C) = \exists R.\pi(\neg D)$;*

Based on this, axioms are transformed as follows.

Definition 2 *(Axiom Transformations) For any ontology O,* $\pi(O)$ *is defined as the set* $\{\pi(\alpha) \mid \alpha$ *is an axiom of O*$\}$*, where* $\pi(\alpha)$ *is the transformation performed on each axiom defined as follows:*

- $\pi(\alpha) = \neg\pi(\neg C_1) \sqsubseteq \pi(C)_2$, *if* $\alpha = C_1 \mapsto C_2$;
- $\pi(\alpha) = \pi(C_1) \sqsubseteq \pi(C)_2$, *if* $\alpha = C_1 \sqsubset C_2$;
- $\pi(\alpha) = \{\pi(C_1) \sqsubseteq \pi(C)_2, \pi(\neg C_2) \sqsubseteq \pi(\neg C_1)\}$, *if* $\alpha = C_1 \to C_2$;.
- $\pi(C(a)) = \pi(C)(a), \pi(R)(a,b) = R(a,b)$,

where a, b *are individuals,* C_1, C_2*,* C *are concepts,* R *a role.*

We note two issues. First, the transformation algorithm is linear in the size of the ontology. Second, for any $\mathcal{ALC}$ ontology O, $\pi(O)$ is still an $\mathcal{ALC}$ ontology. Based on these two observations as well as the following theorem, we can see that paraconsistent reasoning of $\mathcal{ALC}$ can indeed be simulated on standard reasoners by means of the transformation just given.

Theorem 1 *For any ontology O in* $\mathcal{ALC}$ *we have* $O \models_4 \alpha$ *if and only if* $\pi(O) \models_2 \pi(\alpha)$*, where* $\models_2$ *is the entailment in classical* $\mathcal{ALC}$*.*

The following definition, also employed in [13], will be required to ensure that knowledge bases which are inconsistent under the classical semantics become consistent, after transformation, under the four-valued semantics – see Proposition 4.

Definition 3 *Given a knowledge base O, the satisfiable form of O, written SF(O), is a knowledge base obtained by replacing each occurrence of* $\bot$ *in O with* $A_{new} \sqcap \neg A_{new}$*, and replacing each occurrence of* $\top$ *in* (O) *with with* $A_{new} \sqcup \neg A_{new}$*, where* A_{new} *is a new atomic concept.*

3 Paraconsistent Semantics for Expressive DLs

In this section, we study how to extend four-valued semantics to $\mathcal{SROIQ}$. For the conflicting assertion set $\{\geq (n+1)R.C(a), \leq nR.C(a)\}$, intuitively, it is caused by the contradiction that there should be less than n different individuals related to a via the R relation, and also there should be more than $n+1$ different individuals related to a via R. That is, the contradiction is from the set of individuals of concept C which relate a via R. By this idea, we extend the four-valued semantics to the constructors for number restrictions (with four-valued semantics to nominal) in Table 3. We remark that the semantics of roles is just the classical semantics. So the semantics for role inclusion and transitive role axiom are still classical.

We give the following example to illustrate the intuition of our four-valued semantics for number restrictions $\geq nR.C$ given above.

Example 1 *Consider the knowledge base*

$$\{\geq 2hasStu.PhD(\textit{Green}), \leq 1hasStu.PhD(\textit{Green})\}$$

Table 3. Four-valued Semantics Extension to Number Restrictions and Nominals

Constructor	Semantics
$\geq nR.C$	$\langle\{x \mid \#(y.(x,y) \in R^I \wedge y \in \mathit{proj}^+(C^I)) \geq n\},$ $\{x \mid \#(y.(x,y) \in R^I \wedge y \notin \mathit{proj}^-(C^I)) < n\}\rangle$
$\leq nR.C$	$\langle\{x \mid \#(y.(x,y) \in R^I \wedge y \notin \mathit{proj}^-(C^I)) \leq n\},$ $\{x \mid \#(y.(x,y) \in R^I \wedge y \in \mathit{proj}^+(C^I)) > n\}\rangle$
$\{o_1, ...o_n\}$	$\langle\{o_1^I, ..., o_n^I\}, N\rangle$, where $N \subseteq \Delta^I$

which states the conflicting facts that Green has at least two and at most one PhD student. Consider a 4-interpretation $I = (\Delta^I, \cdot^I)$ *where* $\Delta^I = \{a_1, a_2, b_1, b_2, \mathit{Green}\}$, $PhD^I = \langle\{a_1, b_1\}, \{b_1, b_2, a_2\}\rangle$, $hasStu^I = \{(\mathit{Green}, a_1), (\mathit{Green}, a_2), (\mathit{Green}, b_1), (\mathit{Green}, b_2)\}$. *According to Table 3, I is a 4-model because* $(\geq 2hasStu.PhD(\mathit{Green}))^I$ = $(\leq 1hasStu.PhD(\mathit{Green}))^I = B$ *and by checking*

$$\mathit{Green} \in \{x \mid \#(y.(x,y) \in hasStu^I \wedge y \in \mathit{proj}^+(PhD^I)) \geq 2\},$$
$$\mathit{Green} \in \{x \mid \#(y.(x,y) \in hasStu^I \wedge y \notin \mathit{proj}^-(PhD^I)) < 2\}.$$

This example shows that the contradictions on the constructor of number restriction $\geq nR.C$ is reflected by the contradiction on C, which is our underlying idea of Table 3. Generalizing this example, we have the following property which shows that if we have contradictions of the form of $\{\geq nR.C(a), \leq mR.C(a)\} \subseteq O$ with $(m < n)$, then there will be at least $n - m$ individuals relating a via R and contradictorily belonging to concept C under its four-valued model:

Proposition 2 *Given an ontology O, if* $\{\geq nR.C(a), \leq mR.C(a)\} \subseteq O$ *and* $m < n$, *then for any four-valued interpretation I of O, we have*

$$\#\{b \mid (a,b) \in R^I \text{ and } C^I(b) = B\} \geq n - m.$$

Proof. Suppose $C^I = \langle P, N\rangle$, denote $T = \{y \mid (a,y) \in R^I\}$, $T_1 = \{y \mid (a,y) \in R^I \wedge y \in P\}$, $T_1 = \{y \mid (a,y) \in R^I \wedge y \in N\}$. It is equal to prove that $|T_1 \cap T_2| \geq n - m$.

From the assumption and Table 3, we have $a \in \{x \mid \#(y.(x,y) \in R^I \wedge y \in P) \geq n\}$ and $a \in \{x \mid \#(y.(x,y) \in R^I \wedge y \notin N) \leq m\}$. That is, $\#(y.(a,y) \in R^I \wedge y \in P) \geq n$ and $\#(y.(a,y) \in R^I \wedge y \notin N) \leq m$, which means $|T_1| \geq n$ and $|T| - |T_2| \leq m$, Then, it is easy to see that $|T_1| + |T_2| \geq |T| + n - m$. Because $T_1 \subseteq T, T_2 \subseteq T$, we have $|T_1 \cap T_2| \geq n - m$. □

With the following example, we explain the intuition behind our treatment of nominals. Let $O = \{EuropeanState \sqsubseteq \exists currency.\{euro\}, (\forall currency.\neg\{euro\})(UK), EuropeanState(UK)\}$, which states that European countries have Euro as their currency and UK is a European country whose currency is not Euro. We can find a 4-model $I = \langle\Delta^I, \cdot^I\rangle$ for O, with $\Delta^I = \{UnitedKingdom, curreuro\}$ and $UK^I = UnitedKingdom, euro^I = curreuro, EuropeanState^I = \langle\{UnitedKingdom\}, \emptyset\rangle$, $currency^I = \{(UnitedKingdom, curreuro)\}$, and the contradictory $(\{euro\})^I = \langle\{curreuro\}, \{curreuro\}\rangle$. This model says that the currency $curreuro$ belongs to the concept $\{euro\}$ contradictorily. That is, we have conflicting information about the currency of UK, which reflects the contradictory situation described in O.

For the extended four-valued semantics defined in Table 3, we have that the following properties hold as under the classical semantics; proofs can be obtained by carefully checking the definition of four-valued semantics.

Proposition 3 *Let C be a concept and R be an object role name. For any four-valued interpretation I defined satisfying Table 3, we have*

$$(\neg(\leq nR.C))^I =_4 (> nR.C)^I \quad \textit{and} \quad (\neg(\geq nR.C))^I =_4 (< nR.C)^I.$$
$$(\exists R.C)^I =_4 (\geq 1R.C)^I \quad \textit{and} \quad (\forall R.C)^I =_4 (< 1R.\neg C)^I.$$

Propositions 3 shows that many intuitive relations between different concept constructors still hold under the four-valued semantics, which is one of the nice properties of our four-valued semantics for handling inconsistency.

The next proposition shows that our definition of four-valued semantics for $\mathcal{SROIQ}$ is enough to handle inconsistencies in a $\mathcal{SROIQ}$ knowledge base.

Proposition 4 *For any $\mathcal{SROIQ}$ knowledge base O, SF(O) always has at least one 4-valued model, where SF$(\cdot)$ operator is defined in Definition 3.*

Proof. We can prove that *SF*(O) has the following 4-valued model I: $A^I = \langle \Delta^I, \cdot^I \rangle$ for each concept name $A \in \textit{SF}(O)$ and $\textit{proj}^+(R^I) = \Delta^I \times \Delta^I$ for each role name R. We prove this in two steps. First, it is not difficult to see that for any instance $a \in \Delta^I$,

$$a \in \textit{proj}^+((\geq nR.C)^I) = \{x \mid \#(y.(x,y) \in \textit{proj}^+(R^I) \wedge y \in \textit{proj}^+(C^I)) \geq n\} \textit{ and}$$
$$a \in \textit{proj}^-((\geq nR.C)^I) = \{x \mid \#(y.(x,y) \in \textit{proj}^+(R^I) \wedge y \notin \textit{proj}^-(C^I)) < n\}.$$

So $(\geq nR.C)^I = \langle \Delta^I, \Delta^I \rangle$. Similarly, $(\leq nR.C(a))^I = \langle \Delta^I, \Delta^I \rangle$ can be proved.

Secondly, we can easy see that every GCI axiom, in the form of $C(a), R(a,b), C \mapsto D, C \sqsubset D$ or $C \rightarrow D$, is satisfied in I according to Table 2. For every complex role inclusion axiom $R_1 \circ \cdots \circ R_n \sqsubseteq S$, they hold because all roles are interpreted on $\Delta^I \times \Delta^I$. □

Note that unqualified number restrictions, $\geq n.R$ and $\leq n.R$ are special forms of number restrictions because of the equations $\leq n.R =_2 \leq nR.\top$ and $\geq n.R =_2 \geq nR.\top$. However, if we defined the four-valued semantics of $\leq n.R (\geq n.R)$ by the four-valued semantics of $\leq nR.\top (\geq nR.\top)$ defined in Table 3 and Table 1, we would find that $\{\leq n.R(a), \geq n+1.R(a)\}$ is still an unsatisfiable set. This is because the following two inequations cannot hold simultaneously since $\top^I = \langle \Delta^I, \emptyset \rangle$:

$$\#(y.(a,y) \in \textit{proj}(R^I) \wedge y \in \textit{proj}^+(\top^I)) \geq n+1$$
$$\#(y.(a,y) \in \textit{proj}(R^I) \wedge y \notin \textit{proj}^-(\top^I)) \leq n$$

To address this problem, we also adopt the *substitution* defined by Definition 3. By substituting $\top$ by $A_{\textit{new}} \sqcup \neg A_{\textit{new}}$ in $\geq (n+1)R.\top$ and $\leq nR.\top$, we can see that $\{\leq n.R(a), \geq n+1.R(a)\}$ has a four-valued model with $\Delta^I = \{a, b_1, ..., b_{n+1}\}, (a, b_i) \in$

R^I for $1 \leq i \leq n+1$, *and* $A^I_{new} = \langle \Delta^I, \Delta^I \rangle$. By doing this, we get a four-valued model I which pushes the contradiction onto the new atomic concept A_{new}.

The underlying idea of the four-valued semantics for nominals is that if the contradiction occurs on a nominal concept, then we explicitly collect the contradictory individuals into the $proj^-$ part of the four-valued semantics of the nominal such that a four-valued model exists.

Next we study how to extend the reduction algorithm to the case of four-valued semantics of $\mathcal{SROIQ}$.

Definition 4 *(Definition 1 extended) For any given concept C, its transformation $\pi(C)$ is the concept obtained from C by the following inductively defined transformation.*

- *If $C = \geq nR.D$ for D a concept and R a role, then $\pi(C) = \geq nR.\pi(D)$;*
- *If $C = \leq nR.D$ for D a concept and R a role, then $\pi(C) = \leq nR.\neg\pi(\neg D)$;*
- *If $C = \neg(\geq nR.D)$ for D a concept and R a role, then $\pi(C) =< nR.\neg\pi(\neg D)$;*
- *If $C = \neg(\leq nR.D)$ for D a concept and R a role, then $\pi(C) => nR.\pi(D)$;*
- *For nominal $\{o_1, ..., o_n\}$, $\pi(\{o_1, ..., o_n\}) = \{o_1, ..., o_n\}$.*
- *For $\neg\{o_1, ..., o_n\}$, $\pi(\neg\{o_1, ..., o_n\}) = \{o_1, ..., o_n\}'$ which is a new nominal.*

Regarding both the extension of number restrictions and of nominals, the following theorem holds, which lays the theoretical foundation for the algorithm of four-valued semantics for expressive DLs.

Theorem 5 *(Theorem 1 extended) For any ontology O in $\mathcal{SROIQ}$, we have $O \models_4 \alpha$ if and only if $\pi(O) \models_2 \pi(\alpha)$, where $\models_2$ is the entailment in classical $\mathcal{SROIQ}$.*

Proof. By carefully checking the proof of Theorem 1 [13], we find that the decomposability of four-valued semantics to two-valued semantics [15] is key to the claim. It is not difficult to check that the number restriction constructors satisfy the decomposability. □

4 Tractable DLs

The forthcoming revision of the Web Ontology Language features so-called *profiles* which are sublanguages of OWL 2 that have desirable properties like polynomial time complexities [19]. In the following, we examine such tractable languages, more precisely $\mathcal{EL}$++, which corresponds to OWL 2 EL, DL-Lite, which corresponds to OWL 2 QL, and Horn-$\mathcal{SHOIQ}$, which is an extension of OWL 2 RL.

We will see that inconsistencies are also unavoidable in these tractable DLs, therefore we consider how to deal with inconsistencies by our approach. We focus on discussing whether the four-valued semantics can preserve the tractability of these tractable DLs. That is, whether the reduction for computing the four-valued semantics transfers tractable DLs still into tractable DLs. If it does, then we can use the four-valued semantics to deal with inconsistency without having to worry about intractability.

4.1 $\mathcal{EL}$++

We do not consider concrete domains. The syntax definition of $\mathcal{EL}$++ knowledge bases is shown in Table 4. $\mathcal{EL}$++ ontologies may also contain role inclusions (RI) of the form $r_1 \circ \cdots \circ r_k \sqsubseteq r$, where $\circ$ denotes role composition.

Table 4. $\mathcal{EL}$++ and Horn-$\mathcal{SHOIQ}\circ$. The Horn-$\mathcal{SHOIQ}\circ$ normal form used is due to [11].

Language	GCIs	Tractability-preserving Inclusions
$\mathcal{EL}++$	$C \sqsubseteq D$, where $C, D = \top \mid \bot \mid \{a\} \mid C_1 \sqcap C_2 \mid \exists r.C$	internal inclusion (only)
Horn-$\mathcal{SHOIQ}\circ$	$\top \sqsubseteq A$, $A \sqsubseteq \bot$, $A \sqcap A' \sqsubseteq B$, $\exists R.A \sqsubseteq B$, $A \sqsubseteq \exists R.B$, $A \sqsubseteq \forall S.B$, $A \sqsubseteq \geq nR.B$, $A \sqsubseteq \leq 1R.B$.	internal inclusion (only)

It is easy to see that an $\mathcal{EL}$++ knowledge base may be inconsistent if we consider the knowledge base $\{A \sqsubseteq \bot, A(a)\}$[2]. So we still hope that the 4-valued semantics can help us to handle inconsistency in $\mathcal{EL}$++ knowledge bases. However, we will see that we don't have as many choices of class inclusion as in $\mathcal{ALC}$ and $\mathcal{SROIQ}$ if we want to maintain the tractability of the 4-valued entailment relationship of $\mathcal{EL}$++. The analysis is as follows.

Obviously, the concept transformation of Definition 1 performing on an $\mathcal{EL}$++ concept produces an $\mathcal{EL}$++ concept. For the transformation of internal inclusion, each $\mathcal{EL}$++ axiom $C \sqsubset D$ is transformed into $\pi(C) \sqsubseteq \pi(D)$ where $\pi(C)$ and $\pi(D)$ are still $\mathcal{EL}$++ concepts, so that $\pi(C) \sqsubseteq \pi(D)$ is still an $\mathcal{EL}$++ axiom. So internal class inclusion does not destroy the tractability of $\mathcal{EL}$++. This property does not hold for material and strong class inclusions as shown by the following counterexamples: $A \sqcap A' \mapsto B$ and $A \sqcap A' \rightarrow B$. They will be transformed into $\neg(A^- \sqcup A'^-) \sqsubseteq B^+$ and $\{A^+ \sqcap A'^+ \sqsubseteq B^+, B^- \sqsubseteq (A^- \sqcup A'^-)\}$ by Definition 2, which are not within the expressivity of $\mathcal{EL}$++. This is mainly because of no negative constructor in $\mathcal{EL}$++.

For role inclusions in $\mathcal{EL}$++, since there is no negative role constructor which can cause inconsistency, we only need to use the classical interpretation for roles as what we do for $\mathcal{ALC}$. So adaptation of 4-valued semantics does not affect the role inclusions axioms.

Theorem 6 *For any give $\mathcal{EL}$++ ontology O and axiom α, $O \models_4 \alpha$ if and only if $\pi(O) \models_2 \pi(\alpha)$. Moreover, if all of the inclusion axioms in O are interpreted as internal inclusion under its four-valued semantics, then $\pi(O)$ is an $\mathcal{EL}$++ ontology.*

Proof. By Theorem 1, it is obvious that O is 4-satisfiable if and only if $\pi(O)$ is two-valued satisfiable. For any inclusion axiom of O, by definition 1 and definition 4, by

[2] Note that to enable four-valued models on this ontology, we still need to first perform the substitution defined in Definition 3.

induction on the construe of concepts of $\mathcal{EL}$++, we can easily get that for any $\mathcal{EL}$++ concept C, $\pi(C)$ is still an $\mathcal{EL}$++ concept. Because for any internal inclusion axiom $C \sqsubset D$, $\pi(C \sqsubset D) = \pi(C) \sqsubseteq \pi(D)$ is an $\mathcal{EL}$++ axiom since $\pi(C)$ and $\pi(D)$ are $\mathcal{EL}$++ concepts. Therefore, $\pi(O) = \{\pi(C) \sqsubseteq \pi(D) \mid C \sqsubset D \in O\}$ is a classical $\mathcal{EL}$++ ontology. $\square$

4.2 Horn-DLs

We ground our discussion on Horn-$\mathcal{SHOIQ}\circ$ as defined in [11]. Then we will point out that the same conclusion holds for other Horn-DLs, like Horn-$\mathcal{SHOIQ}$ [18], which has tractable data complexiy [9]. We define Horn-$\mathcal{SHOIQ}\circ$ by means of a normal form given in [11], which can be found in Table 4 where A, A', B are concept names.

We can see that all of the Horn-$\mathcal{SHOIQ}\circ$ concept constructors preserve its form under $\pi(\cdot)$ operator except $\leq 1R.B$, because $\pi(\leq 1R.B) =\leq 1R.\neg B^-$ according to Definition 4. To still maintain the concept structure of $\leq 1R.B$ within Horn-$\mathcal{SHOIQ}\circ$, we redefine the $\pi(\cdot)$ as follows

Definition 5 *For any Horn-$\mathcal{SHOIQ}\circ$ concept C, $\pi_{Horn}(C)$ is inductively defined as follows:*

- $\pi_{Horn}(C) = \pi(C)$, *if* $C = \top, A, \bot, A \sqcap A', \exists R.A, \forall S.B, \geq nR.B;$
- $\pi_{Horn}(\leq 1R.B) = \leq 1R.B^=,$ *where* $B^= \sqcap B' \sqsubseteq \bot, B^=$ *is a new concept name*,
- $\pi_{Horn}(\neg(\leq 1R.B)) = \geq 2R.B.$

By Definitions 5 and 2, the transformations for material inclusion axiom $A \mapsto \leq 1R.B$, internal inclusion axiom $A \sqsubset \leq 1R.B$ and strong inclusion axiom $A \rightarrow \leq 1R.B$ are as follows:

$$\pi_{\text{Horn}}(A \mapsto \leq 1R.B) = \{\neg A_- \sqsubseteq \leq 1R.B^=, B^= \sqcap B^- \sqsubseteq \bot\}.$$
$$\pi_{\text{Horn}}(A \sqsubset \leq 1R.B) = \{A \sqsubseteq \leq 1R.B^=, B^= \sqcap B^- \sqsubseteq \bot\}.$$
$$\pi_{\text{Horn}}(A \rightarrow \leq 1R.B) = \{A \sqsubseteq \leq 1R.B^=, B^= \sqcap B^- \sqsubseteq \bot, \geq 2R.B \sqsubseteq A_-\}.$$

Obviously, $\pi_{\text{Horn}}(A \sqsubset \leq 1R.B)$ and $\pi_{\text{Horn}}(A \rightarrow \leq 1R.B)$ are Horn-$\mathcal{SHOIQ}\circ$ ontologies, but $\pi_{\text{Horn}}(A \mapsto \leq 1R.B)$ is not. Actually, for some strong inclusion axiom $C \rightarrow D$ of Horn-$\mathcal{SHOIQ}\circ$4, its transformation $\pi_{\text{Horn}}(C \rightarrow D)$ may be not within the Horn-$\mathcal{SHOIQ}\circ$ language any more. As counterexamples for material inclusion and strong inclusion, just consider again the counterexample used in the $\mathcal{EL}$++ case. The transformed forms $\neg(A^- \sqcup A'^-) \sqsubseteq B^+$ and $\{A^+ \sqcap A'^+ \sqsubseteq B^+, B^- \sqsubseteq (A^- \sqcup A'^-)\}$ are not within the expressivity of Horn-$\mathcal{SHOIQ}\circ$. Since $A \sqcap A' \sqsubseteq B$ is allowed in other Horn-DLs, the same conclusion as for Horn-$\mathcal{SHOIQ}\circ$ holds. This means that when we want ro preserve the structure of tractable Horn-DLs, we have to choose internal inclusion as the only inclusion form to perform paraconsistent reasoning.

Similarly to the case of $\mathcal{EL}$++, we have the following theorem, which guarantees that internal inclusion axiom can preserve the expressivity of Horn-$\mathcal{SHOIQ}\circ$:

Theorem 7 *For any Horn-$\mathcal{SHOIQ}\circ$ ontology O, if any class inclusion axiom in O is interpreted as internal inclusion, then $\pi_{Horn}(O)$ is a Horn-$\mathcal{SHOIQ}\circ$ ontology.*

By Definitions 5 and 2, the conclusion holds obviously. Note that $\pi_{\text{Horn}}(\leq 1R.A)$ contains a Horn-$\mathcal{SHOIQ}\circ$ concept and an additional Horn-$\mathcal{SHOIQ}\circ$ inclusion axiom under classical semantics.

Note that the case just treated covers DLP [4], which corresponds to OWL 2 RL. In particular, the above transformation, properly restricted, shows that DLP transforms into DLP under internal inclusion, so tractability is preserved when the four-valued semantics is applied.

4.3 DL-Lite

The DL-Lite family includes DL-Lite$_{core}$, DL-Lite$_{\mathcal{F}}$, and DL-Lite$_{\mathcal{R}}$; the latter corresponds to OWL 2 QL. The logics of the DL-Lite family are the maximal DLs supporting efficient query answering over large amounts of instances. In [3], the usual DL reasoning tasks on DL-Lite family are shown to be polynomial in the size of the TBox, and query answering is LogSpace in the size of the ABox. Moreover, the DL-Lite family allows for separation between TBox and ABox reasoning during query evaluation: the part of the process requiring TBox reasoning is independent of the ABox, and the part of the process requiring the ABox can be carried out by an SQL engine [3].

Concepts and roles of DL-Lite family are formed by the following syntax [3]:

$$B ::= A \mid \exists R \qquad R ::= P \mid P^-$$
$$C ::= B \mid \neg B \qquad E ::= R \mid \neg R$$

where A denotes an atomic concept, P an atomic role, and P^- the inverse of the atomic role P. See to Table 5 for the syntax definitions of GCIs and Role Inclusions.

Table 5. DL-Lite Family

Language	GCIs	Role Inclusions	Tractability-preserving Inclusions
DL-Lite$_{core}$	$B \sqsubseteq C$	$\emptyset$	internal inclusion (only)
DL-Lite$_{\mathcal{R}}$	$B \sqsubseteq C$	$R \sqsubseteq E$	internal inclusion (only)
DL-Lite$_{\mathcal{F}}$	$B \sqsubseteq C$	(funct R)	internal inclusion (only)

It is also easy to construct an inconsistent knowledge base even for DL-Lite$_{core}$. For instance, $KB = \{B \sqsubseteq \neg A, B(a), A(a)\}$. Moreover, conflictions about roles possibly occur on DL-Lite$_{\mathcal{R}}$, such as $\{P \sqsubseteq P', P \sqsubseteq \neg P', P(a,b)\}$.

In order to still adopt 4-valued semantics for the DL-Lite family, we define the four-valued semantics extension for roles. Just as the four-valued semantics for concepts, a pair $\langle R_P, R_N \rangle$ ($R_P, R_N \subseteq (\Delta^I)^2$) denotes the four-valued semantics of a role R under interpretation I, where R_P stands for the set of pairs of individuals which are related via R and R_N explicitly represents the set of pairs of individuals which are not related via R. Table 6 gives the formal definition.

Table 6. Four-valued Semantics of DL-Lite

Syntax about Roles	**Semantics**
R	$R^I = \langle R_P, R_N \rangle$, where $R_P, R_N \subseteq \Delta^I \times \Delta^I$
R^-	$(R^-)^I = \langle R_P^-, R_N^- \rangle$, where R_P^-, R_N^- represent the inverse relations on R_P^- and R_N^-, respectively.
$\neg R$	$(\neg R)^I = \langle R_N, R_P \rangle$
$\exists R$	$\langle \{x \mid \exists y, (x, y) \in R_P^I\}, \{x \mid \neg\exists y, (x, y) \notin R_N^I\} \rangle$
$\neg\exists R$	$\langle \{x \mid \neg\exists y, (x, y) \notin R_N^I\}, \{x \mid \exists y, (x, y) \in R_P^I\} \rangle$
$=_4$	$(=_4)^I = \langle =_P, =_N \rangle$, where $=_P, =_N \subseteq (\Delta^I)^2$
$(Func\, R)$	for any $x, y, z \in \Delta^I$, if $(x, y) \in R_P$ and $(x, z) \in R_P$, then $(y, z) \in =_P$

For simplifying notation, we say that x and y are *positively related* via R under interpretation I if $(x, y) \in R_P^I$, and that x and y are *negatively related* via R under interpretation I if $(x, y) \in R_N^I$.

Intuitively, the first part of the four-valued semantics $\exists R$ denotes the set of individuals x which have an individual y positively related to x via R. While the second part of the four-valued semantics $\exists R$ in Table 6 denotes the set of individuals x which do not have any individual y that is not negatively related to x via R. Note that x is not negatively related to y does not mean x and y are positively related under the four-valued semantics, since $R_P^I \cup R_N^I = \Delta^I \times \Delta^I$ and $R_P^I \cap R_N^I = \emptyset$ are not necessary to hold under the four-valued semantics. This is also the key point why our four-valued semantics can tolerant conflicts caused by role assertions, by allowing a, b both positively related and negatively related via R under a four-valued interpretation I. Similarly as the four-valued semantics for concepts, by imposing $R_P^I \cup R_N^I = \Delta^I \times \Delta^I$ and $R_P^I \cap R_N^I = \emptyset$ on a four-valued interpretation I, I degenerates into a two-valued interpretation.

By the following example, we can see more clearly the intuition underlying the four-valued semantics of $\exists R$:

Example 2 *Given ontology* $O = \{\, \exists hasStud(Green), \neg\exists hasStud(Green)\}$ *which is inconsistent, consider the following four-valued interpretation* $I = (\Delta^I, \cdot^I)$*, where* $\Delta^I = \{a, b, Green\}$*:*

$$hasStud^I = \langle \{(Green, a)\}, \{(Green, a), (Green, b), (Green, Green)\} \rangle.$$

Under this interpretation, it means that there is information that supports $Green$ *having a student* a*, and there is also information which shows that* $Green$ *does not relate to any individual via role* $hasStudent$*. By checking the following formula and by Table 6, we know that* I *is a 4-model of* O*:*

$$Green \in \{\text{three exists } y \in \Delta^I, \text{ such that } (Green, y) \in hasStud_P^I\}$$
$$Green \in \{\text{for all } y \in \Delta^I, (Green, y) \in hasStud_N^I\}.$$

Intuitively, this 4-model reflects the contradictory situation about the fact of $Green$ *having a student.*

To define a four-valued semantics for DL-Lite$_{\mathcal{F}}$ which can tolerate inconsistency, we need to give a four-valued semantics for equality as shown in Table 6, where we use $=_4$ to emphasize the four-valued semantics version of equality and to distinguish from classical equality $=$, and $=_P$ stands for the set of pairs of equal individuals and $=_N$ for the pairs of inequal individuals. To allow expressing inconsistency, the unique name assumption (UNA) is interpreted as: for any $a, b \in ABox, (a^I, b^I) \in =_N$ for any 4-interpretation I. By $=_4$, we can say that two individuals have the information to be same. Based on this, we can define the four-valued semantics for functionality axioms as shown in Table 6. Then if we have an ontology which contains $\{(Func\ R), R(a,b), R(a,c)\}$ and which is inconsistent under the UNA, it can have a 4-model I by assigning $(a^I, b^I) \in =_P \cap =_N$.

Now we turn to define the concept transformations for DL-Lite.

Definition 6 *The concept and role transformations for DL-Lite concepts are defined by structural induction as follows.*

- *For $E = R$, then $\pi_{Lite}(E) = R$;*
- *For $E = \neg R$, then $\pi_{Lite}(E) = R'$, where R' is a new role;*
- *If $C = \exists R$, then $\pi_{Lite}(C) = \exists R$;*
- *If $C = \neg\exists R$, then $\pi_{Lite}(C) = \neg\exists R^{=}$, where $R^{=}$ is a new role name and $R^{=} \sqsubseteq \neg R'$;*

Considering the internal inclusion transformation, we have that all the GCIs $B \sqsubseteq C$ of DL-Lite will be transferred into the form $B \sqsubseteq C$ with at most an additional role inclusion because $\pi_{\text{Lite}}(B \sqsubseteq \neg\exists R) = \{B \sqsubseteq \neg\exists R^{=}, R^{=} \sqsubseteq \neg R'\}$. For material inclusion and strong inclusion, because the negative concept is not allowed to occur on the left of a GCI, they do not preserve the DL-Lite structure. So only internal inclusion works under the reduction from four-valued semantics to classical semantics of the DL-Lite family to keep tractability.

Again note that $\pi_{\text{Lite}}(B \sqsubseteq \neg\exists R) = \{B \sqsubseteq \neg\exists R^{=}, R^{=} \sqsubseteq \neg R'\}$, which means the transformed ontology may contain a role inclusion axiom. We can see that the four-valued semantics of DL-Lite can be reduced to the reasoning of classical DL-Lite$_{\mathcal{R}}$, as shown by the following theorem.

Theorem 8 *For any ontology DL-Lite O, $O \models_4 \alpha$ if and only if $\pi_{Lite}(O) \models_{\text{DL-Lite}_{\mathcal{R}}} \pi_{Lite}(\alpha)$, where $\models_{\text{DL-Lite}_{\mathcal{R}}}$ is the entailment in classical DL-Lite$_{\mathcal{R}}$.*

Proof. Similarly to the cases of $\mathcal{ALC}$ and $\mathcal{SROIQ}$, by checking the decomposability of four-valued DL-Lite semantics to classical semantics, we can see the theorem hold by noting that the operator $\pi_{\text{Lite}}(\cdot)$ performing on internal axioms which contain $\neg\exists R$ may produce a new role axiom in the form of $R^{=} \sqsubseteq \neg R^c$, which is in the expressivity of DL-Lite$_{\mathcal{R}}$. $\square$

5 Conclusions

In this paper, we extended on our previous study of the four-valued semantics for description logics, and especially adapted it for OWL 2 and its tractable fragments. We formally defined their four-valued semantics and proper reductions to the classical semantics, such that all the benefits from existing reasoners on these DLs can be taken advantage of by invoking classical reasoners after employing the presented reduction algorithms in a preprocessing manner. Furthermore, the preprocessing transformations are linear in the size of the knowledge bases. Unlike the four-valued semantics for $\mathcal{ALC}$ and $\mathcal{SROIQ}$, we showed that in order to preserve the tractability of tractable DLs, only internal class inclusion among the three class inclusion forms is suitable.

Our approach has already been implemented as part of the NeOn Toolkit[3] plugin RaDON. The plugin, which is described in [10], encompasses several methods for inconsistency handling in OWL ontologies. The paraconsistent reasoning algorithm of RaDON, which is based on the work presented in this paper, leaves it to the user to decide how class inclusion axioms are transformed.

Future work on this topic can go into several directions. Adaptation of our approach to obtain tractable paraconsistent reasoning support for larger tractable languages than those presented here, e.g. for ELP [12], will enhance its potential applicability. We also consider it important to investigate on which grounds reasonable choices for transforming inclusion axioms can be made; indeed there may be alternative choices to the three inclusion axioms presented here, which may be useful in certain contexts.

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